Confidential

NEW/OLD

Department of Examination - Sri Lanka

G.C.E. (A/L) Examination - 2020

10 - Combined Mathematics - I

NEW/OLD Syllabus

Marking Scheme

This document has been prepared for the use of Marking Examiners. Some changes would be made according to the views presented at the Chief Examiners' meeting.

Amendments to be included

G.C.E. (A.L.) Examination - 2020

10 - Combined Mathematics I

(New/Old Syllabus)

Distribution of Marks

Paper I

Part A : 10 × 25 = 250

Part B : 05 × 150 = 750

Total = 1000 / 10

Paper I Final Mark = 100

Common Techniques of Marking Answer Scripts.

It is compulsory to adhere to the following standard method in marking answer scripts and entering marks into the mark sheets.

1. Use a red color ball point pen for marking. (Only Chief/Additional Chief Examiner may use a mauve color pen.)
2. Note down Examiner's Code Number and initials on the front page of each answer script.
3. Write off any numerals written wrong with a clear single line and authenticate the alterations with Examiner's initials.
4. Write down marks of each subsection in a $\triangle$ and write the final marks of each question as a rational number in a $\square$ with the question number. Use the column assigned for Examiners to write down marks.

Example: **Question No. 03**

(i) ....
.... √ $\triangle \frac{4}{5}$
....

(ii) ....
.... √ $\triangle \frac{3}{5}$
....

(iii) ....
.... √ $\triangle \frac{3}{5}$
....

(03) (i) $\frac{4}{5}$ + (ii) $\frac{3}{5}$ + (iii) $\frac{3}{5}$ = $\boxed{\frac{10}{15}}$

MCQ answer scripts: (Template)

1. Marking templets for G.C.E.(A/L) and GIT examination will be provided by the Department of Examinations itself. Marking examiners bear the responsibility of using correctly prepared and certified templates.
2. Then, check the answer scripts carefully. If there are more than one or no answers Marked to a certain question write off the options with a line. Sometimes candidates may have erased an option marked previously and selected another option. In such occasions, if the erasure is not clear write off those options too.
3. Place the template on the answer script correctly. Mark the right answers with a 'V' and the wrong answers with a 'X' against the options column. Write down the number of correct answers inside the cage given under each column. Then, add those numbers and write the number of correct answers in the relevant cage.

Structured essay type and assay type answer scripts:

1. Cross off any pages left blank by candidates. Underline wrong or unsuitable answers. Show areas where marks can be offered with check marks.

2. Use the right margin of the overland paper to write down the marks.

3. Write down the marks given for each question against the question number in the relevant cage on the front page in two digits. Selection of questions should be in accordance with the instructions given in the question paper. Mark all answers and transfer the marks to the front page, and write off answers with lower marks if extra questions have been answered against instructions.

4. Add the total carefully and write in the relevant cage on the front page. Turn pages of answer script and add all the marks given for all answers again. Check whether that total tallies with the total marks written on the front page.

Preparation of Mark Sheets.

Except for the subjects with a single question paper, final marks of two papers will not be calculated within the evaluation board this time. Therefore, add separate mark sheets for each of the question paper. Write paper 01 marks in the paper 01 column of the mark sheet and write them in words too. Write paper II Marks in the paper II Column and wright the relevant details. For the subject 51 Art, marks for Papers 01, 02 and 03 should be entered numerically in the mark sheets.

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New Syllabus

1. Using the **Principle of Mathematical Induction**, prove that $\sum_{r=1}^{n}(4r+1) = n(2n+3)$ for all $n \in \mathbb{Z}^+$.

For $n = 1$, L.H.S. = $4 + 1 = 5$ and R.H.S. = $1(2 + 3) = 5$ and hence, L.H.S. = R.H.S.

Hence the result is true for $n = 1$. (5)

Let k be any positive integer and suppose that the result in true for $n = k$.

i.e. $\sum_{r=1}^{k} (4r + 1) = k(2k + 3)$. (5)

$$\text{Now } \sum_{r=1}^{k+1} (4r+1) = \sum_{r=1}^{k} (4r+1) + \{4(k+1)+1\}$$

$$= k(2k+3) + (4k+5) \quad (5)$$

$$= 2k^2 + 7k + 5$$

$$= (k+1)(2k+5) \quad (5)$$

$$= (k+1)[2(k+1)+3]$$

Hence, if the result is true for $n = k$, it is also true for $n = k + 1$. The result is true for $n = 1$ also.

Hence, by the Principle of Mathematical Induction, the result is true for all $n \in \mathbb{Z}^+$. (5)

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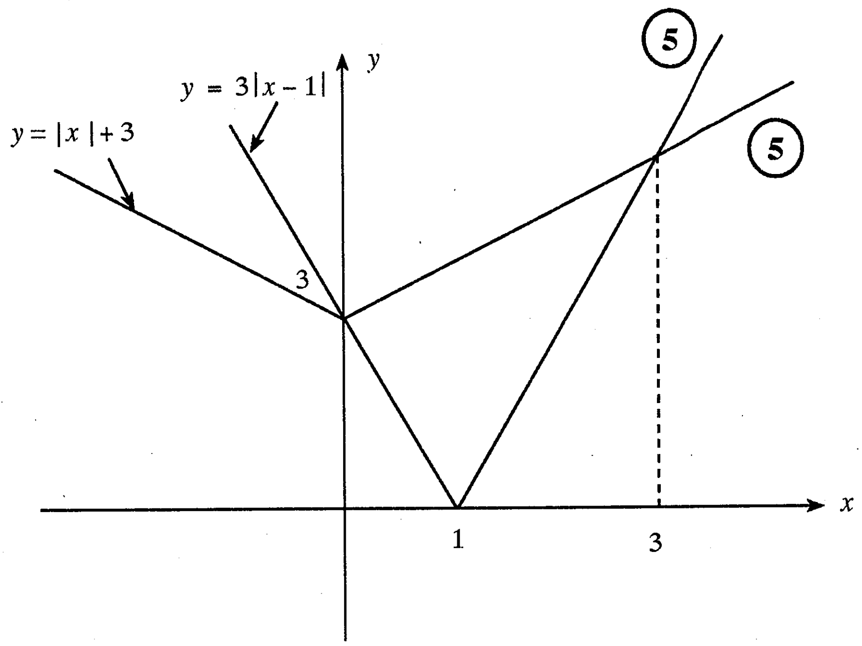
2. Sketch the graphs of $y=3|x-1|$ and $y=|x|+3$ in the same diagram.
Hence or otherwise, find all real values of x satisfying the inequality $3|2x-1|>2|x|+3$.



One point of intersection is given by $x = 0$. The other point of intersection is given by

$$3(x-1) = x+3 \text{ for } x > 1.$$

This gives $x = 3$. (5)

$$3|2x-1| > 2|x| + 3$$

$\Leftrightarrow$ $3|u-1| > |u| + 3$, where $u = 2x$. (5)

$\Leftrightarrow$ $u < 0$ or $u > 3$ (From the graphs)

$\Leftrightarrow$ $x < 0$ or $x > \frac{3}{2}$. (5)

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Aliter 1:

For the graphs (5) + (5), as before.

Aliter for the values of x:

$3\,|2x-1| > 2\,|x|+3$

Case (i) $x \geq \frac{1}{2}$

Then, $3\,|2x-1| > 2\,|x|+3 \Leftrightarrow 3\,(2x-1) > 2x+3$

$\Leftrightarrow 6x-3 > 2x+3$

$\Leftrightarrow x > \frac{3}{2}$

Hence, in this case, the solutions are the values of x satisfying $x > \frac{3}{2}$.

Case (ii) $0 \leq x < \frac{1}{2}$

Then, $3\,|2x-1| > 2\,|x|+3 \Leftrightarrow -6x+3 > 2x+3$

$\Leftrightarrow 0 > 8x$

$\Leftrightarrow 0 > x$

Hence, in this case, there are no solutions.

All 3 cases with correct solutions (10)
Any 2 cases with correct solutions (5)

Case (iii) $x < 0$

Then, $3\,|2x-1| > 2\,|x|+3 \Leftrightarrow -6x+3 > -2x+3$

$\Leftrightarrow 0 > 4x$

$\Leftrightarrow x < 0$

Hence, in this case, the solutions are the values of x satisfying $x < 0$.

Hence, overall the solutions are the values of x satisfying $x < 0$ or $x > \frac{3}{2}$. (5)

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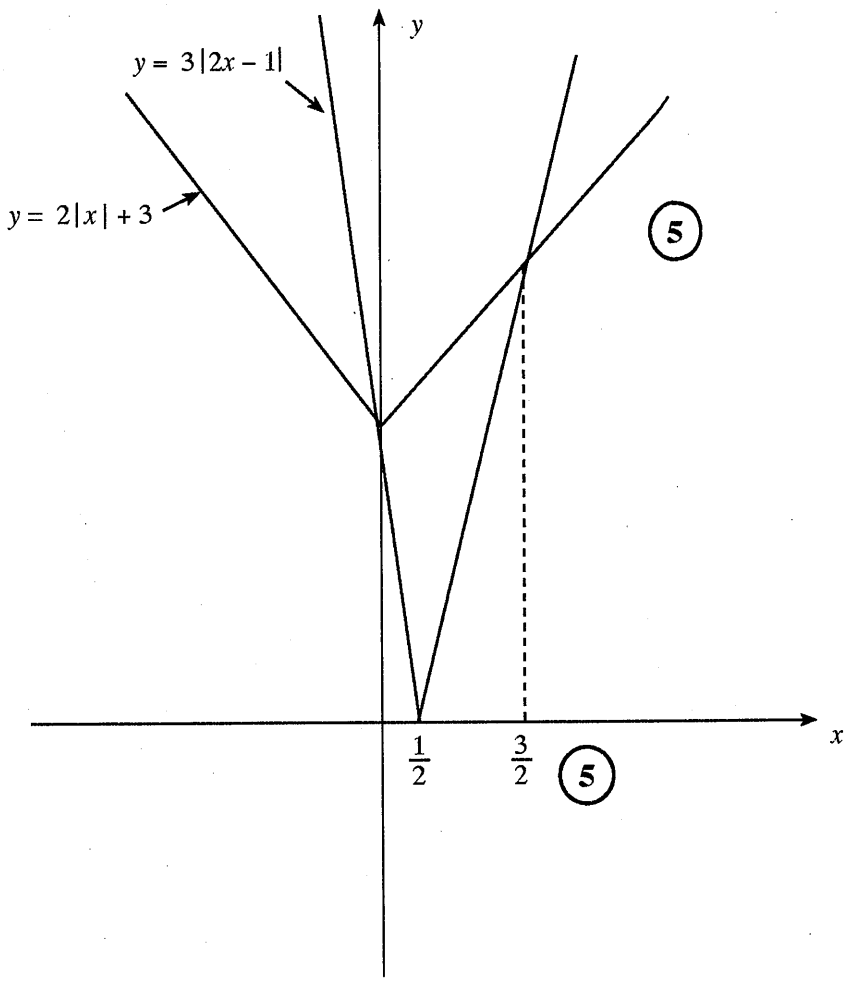
Aliter 2:



From the graphs,

$3|2x-1| > 2|x|+3$

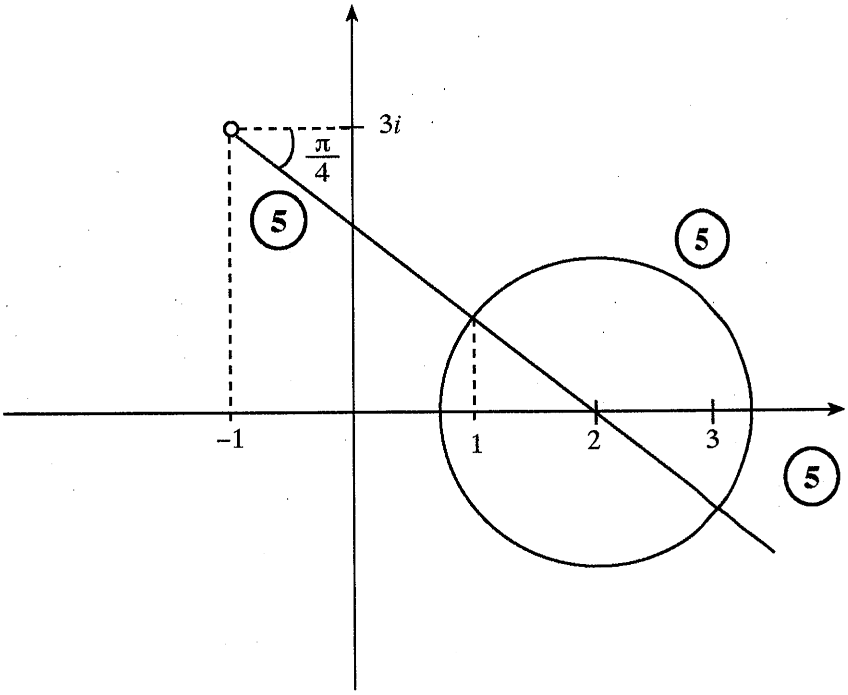
$\Leftrightarrow \quad x < 0$ or $x > \frac{3}{2}$. (5)

3. Sketch, in the same Argand diagram, the loci of the points that represent the complex numbers z satisfying

(i) $\text{Arg}(z+1-3i) = -\frac{\pi}{4}$ and

(ii) $|z-2| = \sqrt{2}$.

Hence, write down the complex numbers represented by the points of intersection of these loci.



The required complex numbers are $1 + i$ (5) and $3 - i$. (5)

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4. Let $n \in \mathbb{Z}^+$. Write down the binomial expansion of $(1+x)^n$ in ascending powers of x.
Show that if the coefficients of two consecutive terms of the above expansion are equal, then n is odd.

$$(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r, \text{ where } {}^nC_r = \frac{n!}{r!\,(n-r)!} \text{ for } r = 1, 2, \ldots, n,$$

(5) (5)

and ${}^nC_0 = 1.$

Two consecutive terms can be taken as

${}^nC_r$ and ${}^nC_{r+1}$

${}^nC_r = {}^nC_{r+1}$ (5) for some $r \in \{0, 1, \ldots\ n-1\}$

$\Leftrightarrow \dfrac{n!}{r!\,(n-r)!} = \dfrac{n!}{(r+1)!\ (n-r-1)!}$ (5)

$\Leftrightarrow \dfrac{1}{n-r} = \dfrac{1}{r+1}$

$\Leftrightarrow n - r = r + 1$

$\Leftrightarrow n = 2r + 1.$

$\therefore n$ is odd. (5)

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Aliter :

Two consecutive terms can be taken as ${}^nC_{r-1}$ and ${}^nC_r$

${}^nC_{r-1} = {}^nC_r$ (5) for some $r \in \{1, 2, 3 \ldots, n\}$

$\Leftrightarrow \dfrac{n!}{[n-(r-1)]!\,(r-1)!} = \dfrac{n!}{(n-r)!\ r!}$ (5)

$\Leftrightarrow \dfrac{1}{n-(r-1)} = \dfrac{1}{r}$

$\Leftrightarrow n - r + 1 = r$

$\Leftrightarrow n = 2r - 1.$

$\therefore n$ is odd. (5)

5. Show that $\lim_{x \to \frac{\pi}{3}} \frac{\sin\left(x - \frac{\pi}{3}\right)}{\left(\sqrt{3x} - \sqrt{\pi}\right)} = \frac{2\sqrt{\pi}}{3}$.

$$\lim_{x \to \frac{\pi}{3}} \frac{\sin(x - \frac{\pi}{3})}{(\sqrt{3x} - \sqrt{\pi})}$$

$$= \lim_{x \to \frac{\pi}{3}} \frac{\sin(x - \frac{\pi}{3})}{(\sqrt{3x} - \sqrt{\pi})} \times \frac{(\sqrt{3x} + \sqrt{\pi})}{(\sqrt{3x} + \sqrt{\pi})} \quad (5)$$

$$= \lim_{x \to \frac{\pi}{3}} \frac{\sin(x - \frac{\pi}{3})}{(3x - \pi)} \cdot (\sqrt{3x} + \sqrt{\pi}) \quad (5)$$

$$= \lim_{x \to \frac{\pi}{3}} \frac{\sin(x - \frac{\pi}{3})}{3(x - \frac{\pi}{3})} \cdot \lim_{x \to \frac{\pi}{3}} (\sqrt{3x} + \sqrt{\pi})$$

$$= \frac{1}{3} \lim_{u \to 0} \frac{\sin u}{u} \quad (5) \quad \cdot (\sqrt{\pi} + \sqrt{\pi}) \quad (5)$$

$$= \frac{1}{3} \cdot 1 \cdot 2\sqrt{\pi} = \frac{2\sqrt{\pi}}{3} \quad (5)$$

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Aliter:

$$\lim_{x \to \frac{\pi}{3}} \frac{\sin(x - \frac{\pi}{3})}{(\sqrt{3x} - \sqrt{\pi})}$$

$$= \lim_{x \to \frac{\pi}{3}} \frac{\sin(x - \frac{\pi}{3})}{(x - \frac{\pi}{3})} \times \frac{(x - \frac{\pi}{3})}{\sqrt{x} - \sqrt{\frac{\pi}{3}}} \times \frac{1}{\sqrt{3}} \quad (5)$$

$$= \left[\lim_{x \to \frac{\pi}{3}} \frac{\sin(x - \frac{\pi}{3})}{(x - \frac{\pi}{3})}\right] \cdot \left[\lim_{x \to \frac{\pi}{3}} \frac{\left(\sqrt{x} - \sqrt{\frac{\pi}{3}}\right)\left(\sqrt{x} + \sqrt{\frac{\pi}{3}}\right)}{\left(\sqrt{x} - \sqrt{\frac{\pi}{3}}\right)}\right] \frac{1}{\sqrt{3}}$$

(5) (5)

$$= 1 \cdot \frac{2\sqrt{\pi}}{3} \cdot \frac{1}{\sqrt{3}} \quad (5)$$

$$= \frac{2\sqrt{\pi}}{3} \quad (5)$$

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6. The region enclosed by the curves $y = \frac{e^x}{1+e^x}$, $x = 0$, $x = \ln 3$ and $y = 0$ is rotated about the x-axis through 2π radians. Show that the volume of the solid thus generated is $\frac{\pi}{4}(4\ln 2 - 1)$.

$$\text{The required volume} = \pi \int_0^{\ln 3} \frac{e^{2x}}{(1+e^x)^2}\, dx \quad (5)$$

$$= \pi \int_2^4 \frac{u-1}{u^2}\, du \qquad \text{Let } u = 1 + e^x. \quad (5)$$

$$= \pi \int_2^4 \left\{\frac{1}{u} - \frac{1}{u^2}\right\} du \quad (5)$$

$$= \pi \left\{\ln |u| + \frac{1}{u}\right\}\Bigg|_2^4 \quad (5)$$

$$= \pi \left\{\ln 4 - \ln 2 + \frac{1}{4} - \frac{1}{2}\right\}$$

$$= \frac{\pi}{4}\{4\ln 2 - 1\} \quad (5)$$

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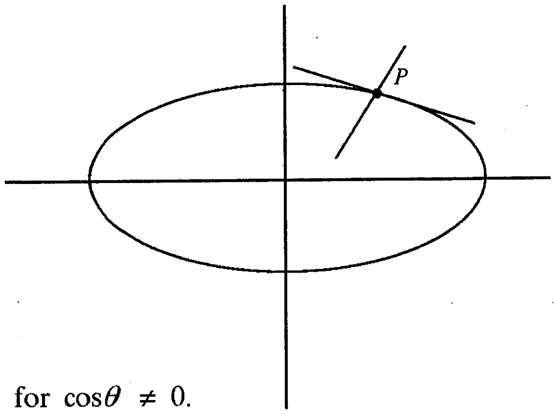
7. Show that the equation of the normal line to the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ at the point $P \equiv (5\cos\theta,\ 3\sin\theta)$ on it, is $5\sin\theta\, x - 3\cos\theta\, y = 16\sin\theta\cos\theta$.

Find the y-intercept of the normal line drawn to the above ellipse at the point $\left(\frac{5}{2}, \frac{3\sqrt{3}}{2}\right)$ on it.



$x = 5\cos\theta,\ y = 3\sin\theta$

$\frac{dx}{d\theta} = -5\sin\theta,\ \frac{dy}{d\theta} = 3\cos\theta$ (5)

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3\cos\theta}{-5\sin\theta}$ (5) for $\sin\theta \neq 0$.

$\therefore$ The gradient of the normal at $P = \frac{5\sin\theta}{3\cos\theta}$ (5) for $\cos\theta \neq 0$.

The required equation is

$$y - 3\sin\theta = \frac{5\sin\theta}{3\cos\theta}(x - 5\cos\theta) \quad \text{for } \cos\theta \neq 0.$$

$$3y\cos\theta - 9\sin\theta\cos\theta = 5x\sin\theta - 25\sin\theta\cos\theta$$

$$5\sin\theta x - 3\cos\theta y = 16\sin\theta\cos\theta. \quad (5)$$

The equation is valid even when $\cos\theta = 0$ (P lies on the y – axis).

For y – intercept : $y = -\frac{16}{3}\sin\theta.$

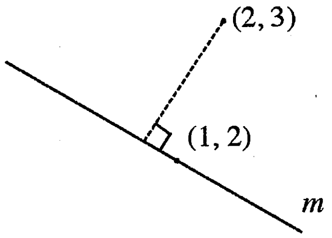
But $3\sin\theta = \frac{3\sqrt{3}}{2} \Rightarrow \sin\theta = \frac{\sqrt{3}}{2}$

$\therefore\ y = -\frac{8}{\sqrt{3}}$ (5)

$\left(0, -\frac{8}{\sqrt{3}}\right)$

25

8. Let $m \in \mathbb{R}$ and l be the straight line passing through the point $A \equiv (1, 2)$ with gradient m. Write down the equation of l in terms of m.
It is given that the perpendicular distance from the point $B \equiv (2, 3)$ to the line l is $\frac{1}{\sqrt{5}}$ units. Find the values of m.



$$y - 2 = m(x - 1)$$

$$y - mx - 2 + m = 0$$ (5)

$$\frac{1}{\sqrt{5}} = \frac{|3 - 2m - 2 + m|}{\sqrt{1 + m^2}}$$ (5)

$$\Leftrightarrow \quad 1 + m^2 = 5(1 - m)^2$$ (5)

$$\Leftrightarrow \quad 1 + m^2 = 5(1 - 2m + m^2)$$

$$\Leftrightarrow \quad 4m^2 - 10m + 4 = 0$$

$$\Leftrightarrow \quad 2m^2 - 5m + 2 = 0$$

$$\Leftrightarrow \quad (2m - 1)(m - 2) = 0$$ (5)

$$\Leftrightarrow \quad m = \frac{1}{2} \text{ or } m = 2.$$

(5)

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9. Find the equation of the circle S having the centre at the point $(-2, 0)$ and passing through the point $(-1, \sqrt{3})$. Write down the equation of the chord of contact of the tangents drawn from the point $A \equiv (1, -1)$ to the circle S.
Hence, show that the x-coordinates of the points of contact of the tangents drawn to S from A satisfies the equation $5x^2 + 8x + 2 = 0$.

$S: (x+2)^2 + y^2 = r^2$ (5)

This goes through $(-1, \sqrt{3})$.

$\therefore \; 1 + 3 = r^2$.

$\therefore \; 4 = r^2$.

Hence, the equation of S is

$(x+2)^2 + y^2 = 4$ (5)

$\therefore \; x^2 + y^2 + 4x = 0$. ———— (1)

The chord of contact of the tangents drawn to S from $A \equiv (1, -1)$ is

$x - y + 2(x+1) = 0$.

i.e. $3x - y + 2 = 0$. (5)

For the points of contact, we substitute $y = 3x + 2$ in (1). (5)

i.e. $x^2 + (3x+2)^2 + 4x = 0$.

Hence, $10x^2 + 12x + 4 + 4x = 0$ and so

$5x^2 + 8x + 2 = 0$. (5)

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10. Let $\theta \neq (2n+1)\frac{\pi}{2}$ for $n \in \mathbb{Z}$.
Using the identity $\cos^2\theta + \sin^2\theta = 1$, show that $\sec^2\theta = 1 + \tan^2\theta$.
It is given that $\sec\theta + \tan\theta = \frac{4}{3}$. **Deduce** that $\sec\theta - \tan\theta = \frac{3}{4}$.
Hence, show that $\cos\theta = \frac{24}{25}$.

$\cos^2\theta + \sin^2\theta = 1$

$\theta \neq (2n+1)\frac{\pi}{2}$ gives us $\cos^2\theta \neq 0$

and hence, $1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$. (5)

$\therefore \sec^2\theta = 1 + \tan^2\theta$. (5)

Now, $\sec^2\theta - \tan^2\theta = 1$ gives us

$(\sec\theta - \tan\theta)(\sec\theta + \tan\theta) = 1$. (5)

Since $\sec\theta + \tan\theta = \frac{3}{4}$, (5)

$\sec\theta - \tan\theta = \frac{4}{3}$.

$\therefore 2\sec\theta = \frac{3}{4} + \frac{4}{3} = \frac{25}{12}$.

$\therefore \cos\theta = \frac{24}{25}$. (5)

25

11. (*a*) Let $f(x)=x^2+px+c$ and $g(x)=2x^2+qx+c$, where $p, q \in \mathbb{R}$ and $c>0$. It is given that $f(x)=0$ and $g(x)=0$ have a common root α. Show that $\alpha=p-q$.

Find c in terms of p and q, and **deduce** that

(i) if $p>0$, then $p<q<2p$,

(ii) the discriminant of $f(x)=0$ is $(3p-2q)^2$.

Let β and γ be the other roots of $f(x)=0$ and $g(x)=0$ respectively. Show that $\beta=2\gamma$.

Also, show that the quadratic equation whose roots are β and γ is given by

$2x^2+3(2p-q)x+(2p-q)^2=0$.

(*b*) Let $h(x)=x^3+ax^2+bx+c$, where $a, b, c \in \mathbb{R}$. It is given that x^2-1 is a factor of $h(x)$. Show that $b=-1$.

It is also given that the remainder when $h(x)$ is divided by x^2-2x is $5x+k$, where $k \in \mathbb{R}$. Find the value of k and show that $h(x)$ can be written in the form $(x-\lambda)^2(x-\mu)$, where $\lambda, \mu \in \mathbb{R}$.

(*a*) Since α is a common root of $f(x) = 0$ and $g(x) = 0$, we have

$\alpha^2+p\alpha+c=0$ —— (1) and (5) $\quad \alpha^2+q\alpha+c=0.$ (5)

$\therefore\ \alpha^2+(q-p)\alpha=0$ (5) and so $\alpha[\alpha-(p-q)]=0$

Hence, $\alpha=p-q.$ (5) $\quad (\because\ c>0 \Rightarrow \alpha \neq 0)$

20

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(1) $\Rightarrow c = -\alpha(\alpha+p)$ (5)

$= -(p-q)(2p-q)$ (5) By substituting for α

$= -(q-p)(q-2p).$

10

(i) $c>0 \Rightarrow (q-p)(q-2p)<0$ (5)

$\Rightarrow q$ lies between p and $2p$.

Assume that $p>0$. Then $p<2p$.

$\therefore\ p<q<2p.$ (5)

10

(ii) $\Delta = p^2 - 4c.$ (5)

$= p^2 + 4(q-p)\ (q-2p)$ (5)

$= p^2 + 4[q^2 - 3pq + 2p^2]$

$= 9p^2 - 12pq + 4p^2$

$= (3p-2q)^2.$ (5)

15

$\alpha + \beta = -p.$ (5)

$\alpha + \gamma = -\dfrac{q}{2}.$ (5)

$\therefore \beta - 2\gamma = -p - \alpha + q + 2\alpha$

$= -p + q + \alpha$

$= 0.$ (5) $(\because \alpha = p - q)$

$\therefore \beta = 2\gamma$

Aliter

$\alpha\beta = c$ (5)

$\alpha r = \dfrac{c}{2}$ (5)

Since $\alpha, \beta, \gamma \neq 0$

$\dfrac{\beta}{\gamma} = 2$ (5)

$\beta = 2\gamma$

15

The required equation is $(x-\beta)\ (x-\gamma) = 0.$

This gives us $x^2 - (\beta+\gamma)x + \gamma\beta = 0.$ (5)

Also, $\beta + \gamma = -p - \dfrac{q}{2} - 2\alpha = -p - \dfrac{q}{2} - (2p-2q) = \dfrac{3}{2}(q-2p).$ (5)

Now, $\alpha^2\beta\gamma = \dfrac{c^2}{2}.$

$\therefore \beta\gamma = \dfrac{c^2}{2(p-q)^2} = \dfrac{(q-p)^2(q-2p)^2}{2(p-q)^2} = \dfrac{1}{2}(q-2p)^2.$ (5)

$x^2 - \dfrac{3}{2}(q-2p)x + \dfrac{1}{2}(q-2p)^2 = 0.$ (5)

$2x^2 + 3(2p-q)x + (2p-q)^2 = 0.$ (5)

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(*b*) Since $(x^2 - 1)$ is a factor of $h(x)$,

$(x-1)$ and $(x+1)$ are both factors of $h(x)$.

Factor theorem gives, $h(1) = 0$ and $h(-1) = 0$. (5)

$h(x) = x^3 + ax^2 + bx + c.$

$\therefore h(1) = 1 + a + b + c = 0$ —— (1) (5) and $h(-1) = -1 + a - b + c = 0.$ —— (2) (5)

By (1) − (2), we get; $2 + 2b = 0.$

$\therefore \; b = -1.$ (5)

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$h(x) = p(x).(x^2 - 2x) + 5x + k$ (5)

$h(0) = k.$ (5)

$\therefore k = c.$

$h(2) = 8 + 4a + 2(-1) + c = 10 + k$ (5)

$4a + c = 4 + k$

$a = 1$ (5)

By (1) + (2), we get; $a = -c.$

$\therefore c = -1.$

Hence, $k = -1.$ (5)

25

$h(x) = x^3 + x^2 - x - 1$

$= (x+1)x^2 - (x+1)$

$= (x+1)(x^2 - 1)$

$= (x+1)^2 (x-1).$ (5)

$\lambda = -1$, $\mu = 1.$ (5)

10

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12. (*a*) It is required to select a musical group consisting of eleven members from among five pianists, five guitarists, three female singers and seven male singers such that it includes **exactly** two pianists and **at least** four guitarists. Find the number of different such musical groups that can be selected.

Find also the number of musical groups among these, having exactly two female singers.

(*b*) Let $U_r = \frac{3r-2}{r(r+1)(r+2)}$ and $V_r = \frac{A}{r+1} - \frac{B}{r}$ for $r \in \mathbb{Z}^+$, where $A, B \in \mathbb{R}$.

Find the values of A and B such that $U_r = V_r - V_{r+1}$ for $r \in \mathbb{Z}^+$.

Hence, show that $\sum_{r=1}^{n} U_r = \frac{n^2}{(n+1)(n+2)}$ for $n \in \mathbb{Z}^+$.

Show that the infinite series $\sum_{r=1}^{\infty} U_r$ is convergent and find its sum.

Now, let $W_r = U_{r+1} - 2U_r$ for $r \in \mathbb{Z}^+$. Show that $\sum_{r=1}^{n} W_r = U_{n+1} - U_1 - \sum_{r=1}^{n} U_r$.

Deduce that the infinite series $\sum_{r=1}^{\infty} W_r$ is convergent and find its sum.

12. (*a*) P = Pianists (5), G = Guitarists (5), Singers (10)

FS - Female Singers (3)

MS - Male Singers (7)

P	G	S	Number of ways
2	4	5	(10) ${}^5C_2\ {}^5C_4\ {}^{10}C_5 = 12600$ (5)
2	5	4	(10) ${}^5C_2\ {}^5C_5\ {}^{10}C_4 = 2100$ (5)

The required number of ways $= 12600 + 2100$

$= 14700$ (5)

35

P	G	FS	MS	Number of ways
2	4	2	3	(10) ${}^{5}C_2\,{}^{5}C_4\,{}^{3}C_2\,{}^{7}C_3 = 5250$ (5)
2	5	2	2	(10) ${}^{5}C_2\,{}^{5}C_5\,{}^{3}C_2\,{}^{7}C_2 = 630$ (5)

The required number of ways $= 5250 + 630$

$= 5880$ (5) **35**

(*b*) For $r \in \mathbb{Z}^+$;

$$U_r = \frac{3r-2}{r(r+1)(r+2)} \quad \text{and} \quad V_r = \frac{A}{(r+1)} - \frac{B}{r}.$$

Thus, $U_r = V_r - V_{r+1}$ gives us $\dfrac{3r-2}{r(r+1)(r+2)} = \dfrac{A}{r+1} - \dfrac{B}{r} - \dfrac{A}{r+2} + \dfrac{B}{r+1}$. (5)

$$\therefore \frac{3r-2}{r(r+1)(r+2)} = \frac{A}{(r+1)(r+2)} - \frac{B}{r(r+1)} \text{ and}$$

hence, $3r - 2 = Ar - B(r+2)$ for $r \in \mathbb{Z}^+$.

(5)

Comparing coefficients of powers of r :

r^1 : $3 = A - B$ ⎫ $A = 4$ (5)

r^0 : $-2 = -2B$ ⎭ $B = 1$ (5)

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$$U_r = V_r - V_{r+1}$$

$r = 1;\quad U_1 = V_1 - V_2$

$r = 2;\quad U_2 = V_2 - V_3$ ⑤

$\vdots \quad \vdots$

$r = n-1;\quad U_{n-1} = V_{n-1} - V_n$

$r = n;\quad U_n = V_n - V_{n+1}$ ⑤

$$\sum_{r=1}^{n} U_r = V_1 - V_{n+1} \quad ⑤$$

$$= 1 - \left(\frac{4}{(n+2)} - \frac{1}{(n+1)}\right) \quad ⑤$$

$$= \frac{n^2}{(n+1)(n+2)}. \quad ⑤$$

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$$\lim_{n\to\infty} \sum_{r=1}^{n} U_r = \lim_{n\to\infty} \left\{\frac{n^2}{(n+1)(n+2)}\right\} \quad ⑤$$

$$= \lim_{n\to\infty} \left\{\frac{1}{(1+\frac{1}{n})\ (1+\frac{2}{n})}\right\}$$

$$= 1. \quad ⑤$$

Therefore, the infinite series $\sum_{r=1}^{\infty} U_r$ is convergent and the sum is 1.

⑤

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$$W_r = U_{r+1} - 2U_r$$

$$\sum_{r=1}^{n} W_r = \sum_{r=1}^{n} (U_{r+1} - 2U_r)$$

$$= \sum_{r=1}^{n} U_r - U_1 + U_{n+1} - 2\sum_{r=1}^{n} U_r \quad ⑤$$

$$= U_{n+1} - U_1 - \sum_{r=1}^{n} U_r. \quad ⑤$$

10

$$\lim_{n \to \infty} \sum_{r=1}^{n} W_r = \lim_{n \to \infty} U_{n+1} - U_1 - \lim_{n \to \infty} \sum_{r=1}^{n} U_r$$

$$= 0 - \frac{1}{6} - 1 \quad (5)$$

$$= -\frac{7}{6}.$$

$\therefore \sum_{r=1}^{\infty} W_r$ is convergent and the sum is $-\frac{7}{6}$. (5)

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13. (*a*) Let $\mathbf{A} = \begin{pmatrix} a+1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a & 2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} a & 1 \\ a & 2 \end{pmatrix}$, where $a \in \mathbb{R}$.

Show that $\mathbf{A}^T\mathbf{B} - \mathbf{I} = \mathbf{C}$; where $\mathbf{I}$ is the identity matrix of order 2.

Show also that $\mathbf{C}^{-1}$ exists **if and only if** $a \neq 0$.

Now, let $a = 1$. Write down $\mathbf{C}^{-1}$.

Find the matrix $\mathbf{P}$ such that $\mathbf{CPC} = 2\mathbf{I} + \mathbf{C}$.

(*b*) Let $z, w \in \mathbb{C}$. Show that $|z|^2 = z\bar{z}$ and applying it to $z - w$,

show that $|z-w|^2 = |z|^2 - 2\,\mathrm{Re}\, z\bar{w} + |w|^2$.

Write a similar expression for $|1-z\bar{w}|^2$ and show that $|z-w|^2 - |1-z\bar{w}|^2 = -(1-|z|^2)(1-|w|^2)$.

Deduce that if $|w| = 1$ and $z \neq w$, then $\left|\dfrac{z-w}{1-z\bar{w}}\right| = 1$.

(*c*) Express $1+\sqrt{3}i$ in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $0 < \theta < \frac{\pi}{2}$.

It is given that $(1+\sqrt{3}i)^m(1-\sqrt{3}i)^n = 2^8$, where m and n are positive integers.

Applying De Moivre's theorem, obtain equations sufficient to determine the values of m and n.

(*a*) $A^TB = \begin{bmatrix} a+1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{2\times 3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ a & 2 \end{bmatrix}_{3\times 2}$ (5)

$= \begin{bmatrix} a+1 & 1 \\ a & 3 \end{bmatrix}$ (5)

$\therefore A^TB - I = \begin{bmatrix} a+1 & 1 \\ a & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (5)

$= \begin{bmatrix} a & 1 \\ a & 2 \end{bmatrix} = C$ (5)

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C^{-1} exists

$\Leftrightarrow |C| \neq 0$ (5)

$\Leftrightarrow 2a - a \neq 0$

$\Leftrightarrow a \neq 0$ (5)

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When $a = 1$, $C = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. (5)

$\therefore\ C^{-1} = \dfrac{1}{2-1} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$. (5)

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$CPC = 2I + C$

$\Leftrightarrow PC = 2C^{-1} + C^{-1}C$ (5)

$\Leftrightarrow PC = 2C^{-1} + I$

$\Leftrightarrow P = 2C^{-1}C^{-1} + C^{-1}$ (5)

$\therefore\ P = 2 \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$= 2 \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ (5)

$= \begin{bmatrix} 10 & -6 \\ -6 & 4 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 12 & -7 \\ -7 & 5 \end{bmatrix}$ (5)

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(*b*) Let $z = x + iy$.

$z\bar{z} = (x + iy)(x - iy)$ (5)

$= x^2 + i^2y^2)$

$= x^2 + y^2$

$= |z|^2$

$\therefore\ |z|^2 = z\bar{z}$. (5)

10

$|z-w|^2$

$= (z-w)\,\overline{(z-w)}$ (5)

$= (z-w)\,(\bar{z}-\bar{w})$ (5)

$= z\bar{z} - z\bar{w} - \bar{z}w + w\bar{w}$

$= |z|^2 - (z\bar{w} + \overline{z\bar{w}}) + |w|^2$ (5)

$= |z|^2 - 2\,\text{Re}\,(z\bar{w}) + |w|^2 \longrightarrow ①$

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$|1 - z\bar{w}|^2$

$= 1 - 2\,\text{Re}\,(z\bar{w}) + |z\bar{w}|^2 \longrightarrow ②$ (5)

05

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① – ② gives;

$|z-w|^2 - |1-z\bar{w}|^2$

$= |z|^2 + |w|^2 - 1 - |z\bar{w}|^2$ (5)

$= -\,(1 - |w|^2 - |z|^2 + |z|^2\,|w|^2)$ (5)

$= -\,(1 - |z|^2)(1 - |w|^2)$ (5)

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$|w| = 1,\ z \neq w$

$\Rightarrow |z-w|^2 - |1 - z\bar{w}|^2 = 0$ (5)

$\Rightarrow |z-w| = |1 - z\bar{w}|$

$\Rightarrow \dfrac{|z-w|}{|1-z\bar{w}|} = 1$ $\left[\begin{array}{l}\because z \neq w \\ \Rightarrow z\bar{w} \neq 1\end{array}\right]$

$\Rightarrow \left|\dfrac{z-w}{1-z\bar{w}}\right| = 1$ (5)

10

(c) $1+\sqrt{3}\,i \;=2\left\{\frac{1}{2}+i\frac{\sqrt{3}}{2}\right\}$ (5)

$=2\left\{\cos\frac{\pi}{3}+i\,\sin\frac{\pi}{3}\right\}$ (5)

10

$(1+\sqrt{3}\,i)^m\,(1-\sqrt{3}\,i)^n=2^m\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^m 2^n\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)^n$ (5)

$=2^{m+n}\left(\cos\frac{m\pi}{3}+i\sin\frac{m\pi}{3}\right)\left(\cos\left(-\frac{n\pi}{3}\right)+i\,\sin\left(-\frac{n\pi}{3}\right)\right)$ (5)

$=2^{m+n}\left(\cos(m-n)\frac{\pi}{3}+i\sin(m-n)\frac{\pi}{3}\right)$ (5)

$\therefore\; 2^{m+n}\left(\cos(m-n)\frac{\pi}{3}+i\sin(m-n)\frac{\pi}{3}\right)=2^8$

$\Rightarrow m+n=8$ and $(m-n)\frac{\pi}{3}=2k\pi\;;\; k\in\mathbb{Z}.$

(5) (5)

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14. (*a*) Let $f(x)=\dfrac{x(2x-3)}{(x-3)^2}$ for $x \neq 3$.

Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x)=\dfrac{9(1-x)}{(x-3)^3}$ for $x \neq 3$.

Hence, find the interval on which $f(x)$ is increasing and the intervals on which $f(x)$ is decreasing.

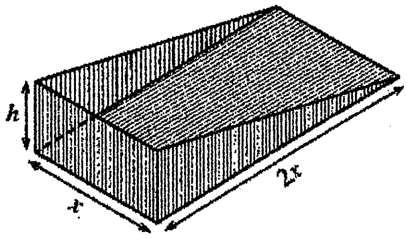
Also, find the coordinates of the turning point of $f(x)$.

It is **given that** $f''(x)=\dfrac{18x}{(x-3)^4}$ for $x \neq 3$.

Find the coordinates of the point of inflection of the graph of $y=f(x)$.

Sketch the graph of $y=f(x)$ indicating the asymptotes, the turning point and the point of inflection.

(*b*) The adjoining figure shows the portion of a dust pan without its handle. Its dimensions in centimetres, are shown in the figure. It is given that its volume x^2h cm^3 is 4500 cm^3. Its surface area S cm^2 is given by $S = 2x^2 + 3xh$. Show that S is minimum when $x = 15$.



(*a*) For $x \neq 3$; $f(x) = \dfrac{x(2x-3)}{(x-3)^2}$

Then, $f'(x) = \dfrac{1}{(x-3)^2}\left[2x-3+2x\right] - \dfrac{2x(2x-3)}{(x-3)^3}$ (20)

$= \dfrac{(x-3)(4x-3)-2x(2x-3)}{(x-3)^3}$

$= \dfrac{4x^2-15x+9-4x^2+6x}{(x-3)^3}$

$= \dfrac{9(1-x)}{(x-3)^3}.$ (5)

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$f'(x) = 0 \Leftrightarrow x = 1.$ (5)

	$-\infty < x < 1$	$1 < x < 3$	$3 < x < \infty$
Sign of $f'(x)$	(–)	(+)	(–)
$f(x)$ is	↘ Decreasing	↗ Increasing	↘ Decreasing
	(5)	(5)	(5)

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Turning point : $\left(1, -\frac{1}{4}\right)$ is a local minimum. (5)

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For $x \neq 3$; $f''(x) = \frac{18x}{(x-3)^4}$.

$f''(x) = 0 \Leftrightarrow x = 0$. (5)

	$-\infty < x < 0$	$0 < x < 3$	$3 < x < \infty$
Sign of $f''(x)$	(−)	(+)	(+)
Concavity	Concave down	Concave up	Concave up

(10)

$\therefore$ Point of inflection = (0, 0). (5)

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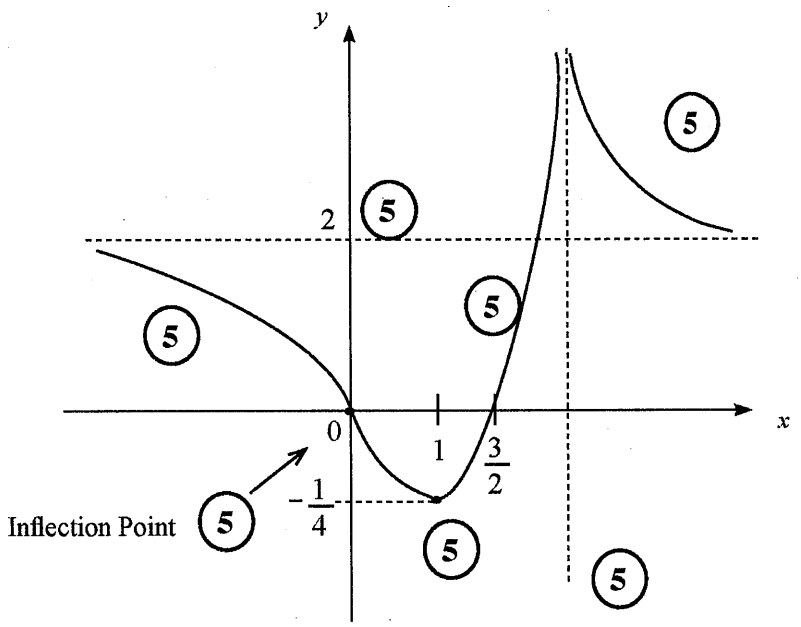
Horizontal asymptote : $\lim_{x \to \pm\infty} f(x) = 2 \quad \therefore y = 2$ (5)

Vertical asymptote : $x = 3$. (5)



45

(**b**) $x^2h = 4500.$

Hence, $S = 2x^2 + 3xh$

$$= 2x^2 + 3x \cdot \frac{4500}{x^2} \quad \text{for } x > 0.$$ (5)

$$\therefore \frac{dS}{dx} = 4x - 3 \times 4500\left(\frac{1}{x^2}\right) = \frac{4(x^3 - 3375)}{x^2}.$$ (5)

$\frac{dS}{dx} = 0$ (5) $\Leftrightarrow x = 15.$ (5)

For $0 < x < 15,\ \frac{ds}{dr} < 0$ (5) and for $x > 15,\ \frac{ds}{dr} > 0.$ (5)

$\therefore S$ is minimum when $x = 15.$ (5)

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15.(*a*) It is given that there exist constants A and B such that

$x^3 + 13x - 16 = A(x^2+9)(x+1) + B(x^2+9) + 2(x+1)^2$ for all $x \in \mathbb{R}$.

Find the values of A and B.

Hence, write down $\dfrac{x^3+13x-16}{(x+1)^2(x^2+9)}$ in partial fractions and

find $\displaystyle\int \frac{x^3+13x-16}{(x+1)^2(x^2+9)}\,dx$.

(*b*) Using integration by parts, evaluate $\displaystyle\int_0^1 e^x \sin^2 \pi x\,dx$.

(*c*) Using the formula $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, where a is a constant,

show that $\displaystyle\int_0^\pi x\cos^6 x \sin^3 x\,dx = \frac{\pi}{2}\int_0^\pi \cos^6 x \sin^3 x\,dx$.

Hence, show that $\displaystyle\int_0^\pi x\cos^6 x \sin^3 x\,dx = \frac{2\pi}{63}$.

(*a*) All $x \in \mathbb{R}$

$x^2 + 13x - 16 = A(x^2+9)(x+1) + B(x^2+9) + 2(x+1)^2$

Comparing coefficients of powers of x ;

x^3 : $1 = A$. (5)

x^0 : $-16 = 9A + 9B + 2 \Rightarrow B = -3$.

(5) (5)

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$$\therefore \frac{x^2+13x-16}{(x+1)^2(x^2+9)} = \frac{1}{(x+1)} - \frac{3}{(x+1)^2} + \frac{2}{x^2+9}. \quad (10)$$

$$\int \frac{x^2+13x-16}{(x+1)^2(x^2+9)}\,dx = \int \frac{1}{x+1}\,dx - 3\int \frac{1}{(x+1)^2}\,dx + 2\int \frac{1}{x^2+9}\,dx$$

$$= \ln|x+1| + \frac{3}{x+1} + \frac{2}{3}\tan^{-1}\left(\frac{x}{3}\right) + C.$$

(5) (5) (5) (5)

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(b)

$$\int_0^1 e^x \sin^2 \pi x \; dx = \frac{1}{2}\int_0^1 e^x (1 - \cos 2\pi x) \; dx \quad (5)$$

$$= \frac{1}{2} e^x \Big|_0^1 - \frac{1}{2} \underbrace{\int_0^1 e^x \cos 2\pi x \; dx}_{I} \quad (5)$$

$$= \frac{1}{2}(e-1) - \frac{1}{2} I . \quad (5) \qquad \text{——— } (1)$$

Now,

$$I = \int_0^1 e^x \cos 2\pi x \quad dx$$

$$= \underbrace{e^x \frac{\sin 2\pi x}{2\pi} \Big|_0^1}_{(5)} - \frac{1}{2\pi} \int_0^1 e^x \sin 2\pi x \; dx \quad (5)$$

$$= \underset{(5)}{0} - \frac{1}{2\pi}\left[\left(- e^x \frac{\cos 2\pi x}{2\pi}\right)\Big|_0^1 + \frac{1}{2\pi} \underbrace{\int_0^1 e^x \cos 2\pi x \; dx}_{I} \right] \quad (5)$$

$$= \frac{1}{4\pi^2}\left[e - 1\right] - \frac{1}{4\pi^2} I . \quad (5) \quad (5)$$

$$\therefore I \left(1 + \frac{1}{4\pi^2}\right) = \frac{1}{4\pi^2}(e-1).$$

$$\therefore I = \frac{(e-1)}{4\pi^2 + 1} . \quad (5)$$

$$\therefore \text{ By } (1), \int_0^1 e^x \sin^2 \pi x \; dx \; \frac{1}{2}(e-1) - \frac{1}{2}\frac{(e-1)}{(4\pi^2+1)} \quad (5)$$

$$= \frac{(e-1)}{2}\left[\frac{4\pi^2}{4\pi^2 + 1}\right]$$

$$= \frac{2(e-1)\pi^2}{1 + 4\pi^2} . \quad (5)$$

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(c) $I = \int_0^{\pi} x\cos^6 x \sin^3 x \, dx$

$= \int_0^{\pi} (\pi - x) \underbrace{\cos^6(\pi - x)}_{\cos^6 x} \underbrace{\sin^3(\pi - x)}_{\sin^3 x} \, dx = \int_0^{\pi} (\pi - x)\cos^6 x \sin^3 x$ (5) (5)

$= \pi \int_0^{\pi} \cos^6 x \sin^3 x \, dx - \underbrace{\int_0^{\pi} x\cos^6 x \sin^3 x \, dx}_{I}$ (5)

$\therefore I = \frac{\pi}{2} \int_0^{\pi} \cos^6 x \sin^3 x \, dx$. (5)

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$I = \frac{\pi}{2} \int_0^{\pi} \cos^6 x \sin^3 x \, dx$

$= \frac{\pi}{2} \int_0^{\pi} \cos^6 x \sin^2 x \sin x \, dx$

$= \frac{\pi}{2} \int_0^{\pi} \cos^6 x (1 - \cos^2 x) \sin x \, dx$ (5)

$= \frac{\pi}{2} \left[\int_0^{\pi} \cos^6 x \sin x \, dx - \int_0^{\pi} \cos^8 x \sin x \, dx \right]$ (5)

$= \frac{\pi}{2} \left[\frac{-\cos^7 x}{7} \Big|_0^{\pi} + \frac{\cos^9 x}{9} \Big|_0^{\pi} \right]$ (5) (5)

$= \frac{\pi}{2} \left[\frac{2}{7} - \frac{2}{9} \right]$ (5)

$= \frac{2\pi}{63}$.

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16. Let $A \equiv (1,2)$ and $B \equiv (3,3)$.

Find the equation of the straight line l passing through the points A and B.

Find the equations of the straight lines l_1 and l_2 passing through A, each making an acute angle $\frac{\pi}{4}$ with l.

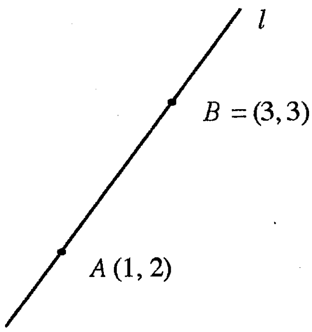
Show that the coordinates of any point on l can be written in the form $(1+2t, 2+t)$, where $t \in \mathbb{R}$.

Show also that the equation of the circle C_1 lying entirely in the first quadrant with radius $\frac{\sqrt{10}}{2}$, touching both l_1 and l_2, and its centre on l is $x^2+y^2-6x-6y+\frac{31}{2}=0$.

Write down the equation of the circle C_2 whose ends of a diameter are A and B.

Determine whether the circles C_1 and C_2 intersect orthogonally.

(16)



gradient $= \frac{3-2}{3-1} = \frac{1}{2}$ (5)

Equation of l: $y-2 = \frac{1}{2}(x-1)$ (5)

i.e. $2y-4 = x-1$

i.e. $x-2y+3 = 0$ (10)

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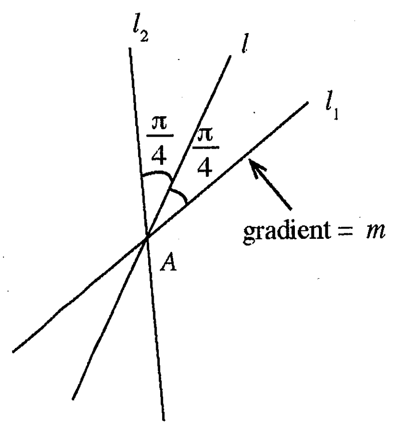

$$\tan\frac{\pi}{4} = \left|\frac{m-\frac{1}{2}}{1+\frac{1}{2}m}\right|$$ (10)

$$\therefore \quad 1 = \left|\frac{2m-1}{2+m}\right|$$ (5)

$\Leftrightarrow \quad 2+m = \pm(2m-1)$

$\Leftrightarrow \quad 2+m = 2m-1$ or $2+m = -2m+1$

$\Leftrightarrow \quad m = 3$ or $m = -\frac{1}{3}$.

(5) (5)

$l_1 : y-2 = 3(x-1)$ and $l_2 : y-2 = -\frac{1}{3}(x-1)$.

$l_1 : 3x - y - 1 = 0$ (5) and $l_2 : x + 3y - 7 = 0$. (5)

or vice versa. **35**

$l : \frac{x-1}{2} = \frac{y-2}{1} = t$ (say). (5)

Then $x = 1+2t,\ y = 2+t$, where $t \in \mathbb{R}$. (5) **10**

For C_1,

The perpendicular distance to l_1 from $P \equiv (1+2t, 2+t)$ is equal to the radius of C_1

i.e. $\frac{|3(1+2t) - (2+t) - 1|}{\sqrt{3^2 + (-1)^2}} = \frac{\sqrt{10}}{2}$. (10) (5)

i.e. $|3 + 6t - 2 - t - 1| = 5.$ (5)

$|5t| = 5.$

$t = \pm 1$ (5)

$P \equiv (3, 3) = B$, since $P \equiv (-1, 1)$ is not suitable. (5) (5)

$C_1 : (x-3)^2 + (y-3)^2 = \frac{5}{2}$. (5)

i.e. $x^2 + y^2 - 6x - 6y + 18 = \frac{5}{2}$

i.e. $x^2 + y^2 - 6x - 6y + \frac{31}{2} = 0$ (5) **45**

The equation of C_2 is

$(x-1)(x-3) + (y-2)(y-3) = 0.$ (10)

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$$2g_1g_2 + 2f_1f_2 = 2(-3)(-2) + 2(-3)\left(-\frac{5}{2}\right) = 27.$$ (10) (5)

$$c_1 + c_2 = \frac{31}{2} + 9 = \frac{49}{2}.$$ (5)

$\therefore\ 2g_1g_2 + 2f_1f_2 \neq c_1 + c_2.$ (5)

$\therefore$ C_1 and C_2 do not intersect orthogonally. (5)

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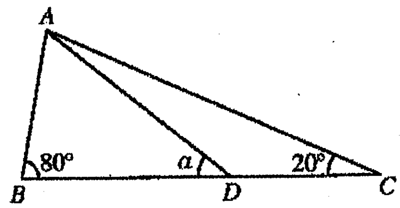
17.(*a*) Write down $\sin(A-B)$ in terms of $\sin A$, $\cos A$, $\sin B$ and $\cos B$.

Deduce that

(i) $\sin(90^\circ-\theta) = \cos\theta$, and

(ii) $2\sin 10^\circ = \cos 20^\circ - \sqrt{3}\sin 20^\circ$.

(*b*) In the usual notation, state the **Sine Rule** for a triangle ABC.



In the triangle ABC shown in the figure, $A\hat{B}C = 80^\circ$ and $A\hat{C}B = 20^\circ$. The point D lies on BC such that $AB = DC$. Let $A\hat{D}B = \alpha$.

Using the **Sine Rule** for suitable triangles, show that $\sin 80^\circ \sin(\alpha - 20^\circ) = \sin 20^\circ \sin\alpha$.

Explain why $\sin 80^\circ = \cos 10^\circ$ and **hence**, show that $\tan\alpha = \dfrac{\sin 20^\circ}{\cos 20^\circ - 2\sin 10^\circ}$.

Using the result in (*a*)(ii) above, **deduce** that $\alpha = 30^\circ$.

(*c*) Solve the equation $\tan^{-1}(\cos^2 x) + \tan^{-1}(\sin x) = \frac{\pi}{4}$.

(*a*) $\sin(A-B) = \sin A\cos B - \cos A \sin B$. (10) **10**

(i) $\sin(90^\circ - \theta) = \sin 90^\circ \cos\theta - \cos 90^\circ \sin\theta$ (5)

$= \cos\theta$. (5) $\quad (\because \sin 90^\circ = 1 \text{ and } \cos 90^\circ = 0.)$ **10**

(ii) $2\sin 10^\circ = 2\sin(30^\circ - 20^\circ)$ (5)

$= 2\sin 30^\circ \cos 20^\circ - 2\cos 30^\circ \sin 20^\circ$ (5)

$= \cos 20^\circ - \sqrt{3}\sin 20^\circ$. (5) $\quad (\because \sin 30^\circ = \frac{1}{2} \text{ and } \cos 30^\circ = \frac{\sqrt{3}}{2})$

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(b) $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{}$, (10)

where $BC = a$, $CA = b$ and $AB = c$. **10**

Using the sine Rule :

for the trangle ABD; $\dfrac{AB}{\sin \alpha} = \dfrac{AD}{\sin 80^\circ}$. (10)

for the trangle ADC; $\dfrac{DC}{\sin(\alpha - 20^\circ)} = \dfrac{AD}{\sin 20^\circ}$. (5)

$\therefore \dfrac{\sin(\alpha - 20^\circ)}{\sin \alpha} = \dfrac{\sin 20^\circ}{\sin 80^\circ}$. (5)

$\therefore \sin 80^\circ \sin(\alpha - 20^\circ) = \sin 20^\circ \sin \alpha$. (5)

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$\sin 80^\circ = \sin(90^\circ - 10^\circ) = \cos 10^\circ$ (5) (5)

Now, $\sin 80^\circ \sin(\alpha - 20^\circ) = \sin 20^\circ \sin \alpha$ gives

$\cos 10^\circ \sin(\alpha - 20^\circ) = 2 \sin 10^\circ \cos 10^\circ \sin \alpha$. (5) (5)

$\therefore \sin\alpha \cos 20^\circ - \cos\alpha \sin 20^\circ = 2 \sin 10^\circ \sin \alpha$. (5)

$\therefore \tan \alpha (\cos 20^\circ - 2 \sin 10^\circ) = \sin 20^\circ$ (5) and hence $\tan \alpha = \dfrac{\sin 20^\circ}{\cos 20^\circ - 2 \sin 10^\circ}$. (5)

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By (a)(ii), we get $\tan\alpha = \dfrac{\sin 20^\circ}{\sqrt{3}\,\sin 20^\circ} = \dfrac{1}{\sqrt{3}}$. (5)

$\therefore\ \alpha = 30^\circ$. (5) $(20^\circ < \alpha < 90^\circ)$

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(*c*) $\tan^{-1}(\cos^2 x) + \tan^{-1}(\sin x) = \dfrac{\pi}{4}$.

Let $\alpha = \tan^{-1}(\cos^2 x)$ and $\beta = \tan^{-1}(\sin x)$

Then, $\alpha = \dfrac{\pi}{4} - \beta$.

$\therefore\ \tan\alpha = \tan\left(\dfrac{\pi}{4} - \beta\right)$ (5)

$= \dfrac{1 - \tan\beta}{1 + \tan\frac{\pi}{4}\ \tan\beta}$. (5)

$\Rightarrow \cos^2 x = \dfrac{1 - \sin x}{1 + \sin x}$. (5)

$\cos^2 x\ (1 + \sin x) = (1 - \sin x)$

$(1 - \sin^2 x)(1 + \sin x) = (1 - \sin x)$ (5)

$(1 - \sin x)(1 + \sin x)^2 = 1 - \sin x$

$\Rightarrow \sin x = 1$ or $1 + \sin x = \pm 1$

$\Rightarrow \sin x = 1$ or $\sin x = 0$ (5) $(\because \sin x \neq -2)$

$\Rightarrow x = n\pi + (-1)^n \dfrac{\pi}{2}$ for $n \in \mathbb{Z}$ (5) or $x = m\pi$ for $m \in \mathbb{Z}$ (5)

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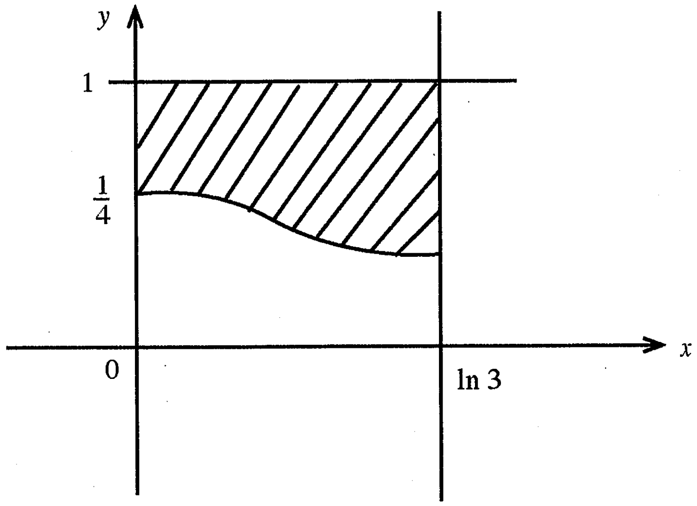
Old Syllabus

6. Show that the area of the region bounded by the curves $y = \frac{e^{2x}}{(1+e^x)^2}$, $x = 0$, $x = \ln 3$ and $y = 1$ is $\ln\left(\frac{3}{2}\right) + \frac{1}{4}$.



(5)

$$\text{Area} = \pi \int_0^{\ln 3} \left\{ 1 - \frac{e^{2x}}{(1+e^x)^2} \right\} dx$$

$$= \ln 3 - \int_2^4 \frac{u-1}{u^2}\, du \qquad (u = 1 + e^x.)$$

(5)

$$= \ln 3 - \int_2^4 \left\{ \frac{1}{u} - \frac{1}{u^2} \right\} du \quad (5)$$

$$= \ln 3 - \left\{ \ln|u| + \frac{1}{u} \right\} \Big|_2^4$$

(5)

$$= \ln 3 - \left\{ \ln 4 - \ln 2 + \frac{1}{4} - \frac{1}{2} \right\}$$

$$= \ln 3 - \left\{ \ln 2 - \frac{1}{4} \right\}$$

$$= \ln\left(\frac{3}{2}\right) + \frac{1}{4}. \quad (5)$$

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7. A curve C is given parametrically by $x = 2t - \cos 2t$ and $y = 1 - \sin 2t$ for $-\frac{\pi}{4} < t < \frac{3\pi}{4}$. Find $\frac{dy}{dx}$ in terms of t.
Show that the equation of the normal line drawn to the curve C at the point on it corresponding to $t = \frac{\pi}{12}$ is $6\sqrt{3}x - 6y - \sqrt{3}\pi + 12 = 0$.

$x = 2t - \cos 2t, \quad y = 1 - \sin 2t$

$\frac{dx}{dt} = 2 + 2\sin 2t, \quad \frac{dy}{dt} = -2\cos 2t.$ (5)

$\frac{dy}{dt} = \frac{-2\cos 2t}{2 + 2\sin 2t} = -\frac{\cos 2t}{1 + \sin 2t}$ (5)

$t = \frac{\pi}{12}$ gives us $x = \frac{\pi}{6} - \frac{\sqrt{3}}{2}$ and $y = 1 - \frac{1}{2} = \frac{1}{2}$. (5)

The gradient of the required normal $= \dfrac{1 + \sin\frac{\pi}{6}}{\cos\frac{\pi}{6}}$

$= \dfrac{\frac{3}{2}}{\frac{\sqrt{3}}{2}} = \sqrt{3}$ (5)

The required equation :

$$y - \frac{1}{2} = \sqrt{3}\left(x - \frac{\pi}{6} + \frac{\sqrt{3}}{2}\right)$$

i.e. $6y - 3 = 6\sqrt{3}x - \sqrt{3}\pi + 9$

i.e. $6\sqrt{3}x - 6y - \sqrt{3}\pi + 12 = 0.$ (5)

13. (*a*) Let $\mathbf{A} = \begin{pmatrix} a+1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a & 2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} a & 1 \\ a & 2 \end{pmatrix}$, where $a \in \mathbb{R}$.

Show that $\mathbf{A}^T\mathbf{B} - \mathbf{I} = \mathbf{C}$; where $\mathbf{I}$ is the identity matrix of order 2.

Show also that $\mathbf{C}^{-1}$ exists **if and only if** $a \neq 0$.

Now, let $a = 1$. Write down $\mathbf{C}^{-1}$.

Find the matrix $\mathbf{P}$ such that $\mathbf{CPC} = 2\mathbf{I} + \mathbf{C}$.

(*b*) Let $z, w \in \mathbb{C}$. Show that $|z|^2 = z\bar{z}$ and applying it to $z - w$,

show that $|z-w|^2 = |z|^2 - 2\,\mathrm{Re}\, z\bar{w} + |w|^2$.

Write a similar expression for $|1 - z\bar{w}|^2$ and show that $|z-w|^2 - |1-z\bar{w}|^2 = -\left(1-|z|^2\right)\left(1-|w|^2\right)$.

Deduce that if $|w| = 1$ and $z \neq w$, then $\left|\dfrac{z-w}{1-z\bar{w}}\right| = 1$.

(*c*) Express $1+\sqrt{3}i$ in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $0 < \theta < \frac{\pi}{2}$.

In an Argand diagram, point O represents the origin and point A represents the complex number $1+\sqrt{3}i$. Let $OABCDE$ be the regular hexagon having O and A as two of its consecutive vertices and the order of vertices taken in the counter clockwise sense. Find the complex numbers represented by the points B, C, D and E.

(*a*) $A^TB = \begin{bmatrix} a+1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{2\times 3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ a & 2 \end{bmatrix}_{3\times 2}$ (5)

$= \begin{bmatrix} a+1 & 1 \\ a & 3 \end{bmatrix}$ (5)

$\therefore\ A^TB - I = \begin{bmatrix} a+1 & 1 \\ a & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (5)

$= \begin{bmatrix} a & 1 \\ a & 2 \end{bmatrix} = C$ (5)

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C^{-1} exists

$\Leftrightarrow |C| \neq 0$ (5)

$\Leftrightarrow 2a - a \neq 0$

$\Leftrightarrow a \neq 0$ (5)

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When $a = 1$, $C = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. (5)

$\therefore\ C^{-1} = \frac{1}{2-1}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$. (5)

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$CPC = 2I + C$

$\Leftrightarrow PC = 2C^{-1} + C^{-1}C$ (5)

$\Leftrightarrow PC = 2C^{-1} + I$

$\Leftrightarrow P = 2C^{-1}C^{-1} + C^{-1}$ (5)

$\therefore\ P = 2\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$= 2\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ (5)

$= \begin{bmatrix} 10 & -6 \\ -6 & 4 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 12 & -7 \\ -7 & 5 \end{bmatrix}$ (5)

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(*b*) Let $z = x + iy$.

$z\bar{z} = (x + iy)(x - iy)$ (5)

$= x^2 + i^2y^2)$

$= x^2 + y^2$

$= |z|^2$

$\therefore\ |z|^2 = z\bar{z}$. (5)

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$|z-w|^2$

$= (z-w)\ \overline{(z-w)}$ (5)

$= (z-w)\ (\overline{z}-\overline{w})$ (5)

$= z\overline{z} - z\overline{w} - \overline{z}w + w\overline{w}$

$= |z|^2 - (z\overline{w} + \overline{z\overline{w}}) + |w|^2$ (5)

$= |z|^2 - 2\,\text{Re}\,(z\overline{w}) + |w|^2 \longrightarrow (1)$

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$|1-z\overline{w}|^2$

$= 1 - 2\,\text{Re}\,(z\overline{w}) + |z\overline{w}|^2 \longrightarrow (2)$ (5)

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(1) – (2) gives;

$|z-w|^2 - |1-z\overline{w}|^2$

$= |z|^2 + |w|^2 - 1 - |z\overline{w}|^2$ (5)

$= -(1 - |w|^2 - |z|^2 + |z|^2\,|w|^2)$ (5)

$= -(1-|z|^2)(1-|w|^2)$ (5)

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$|w| = 1,\ z \neq w$

$\Rightarrow |z-w|^2 - |1-\overline{z}w|^2 = 0$ (5)

$\Rightarrow |z-w| = |1-\overline{z}w|$

$\Rightarrow \dfrac{|z-w|}{|1-\overline{z}w|} = 1$ $\left[\begin{array}{l}\because z \neq w \\ \Rightarrow \overline{z}w \neq 1\end{array}\right]$

$\Rightarrow \left|\dfrac{z-w}{1-\overline{z}w}\right| = 1$ (5)

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(*c*) $1+\sqrt{3}\,i = 2\left\{\frac{1}{2} + i\frac{\sqrt{3}}{2}\right\}$ (5)

$= 2\left\{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right\}$. (5)

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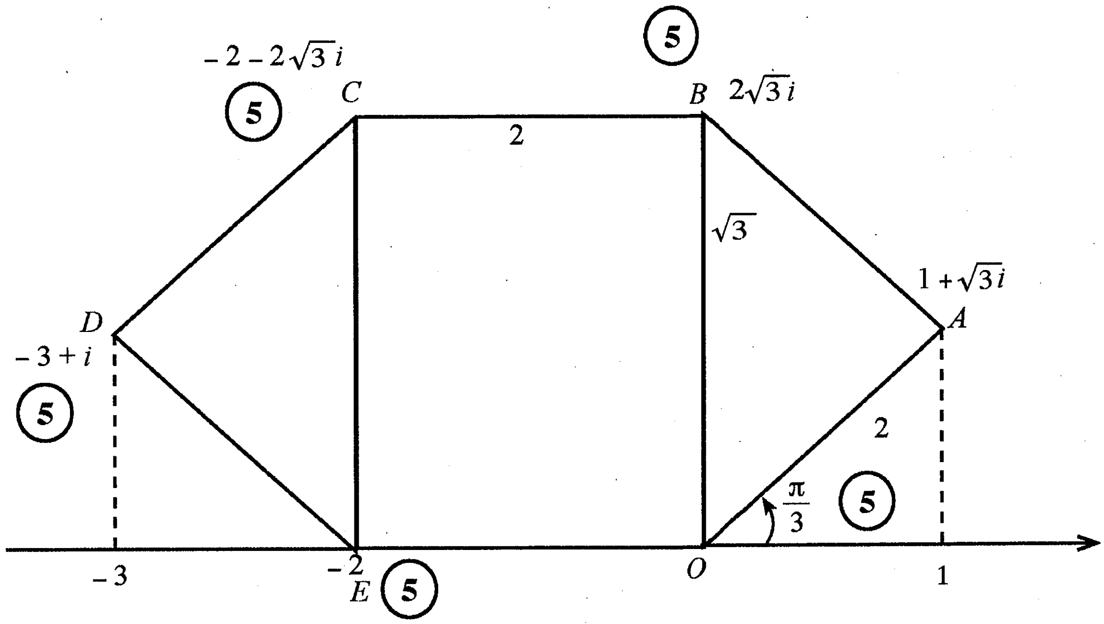

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14. (*a*) Let $f(x)=\dfrac{x(2x-3)}{(x-3)^2}$ for $x\neq 3$.

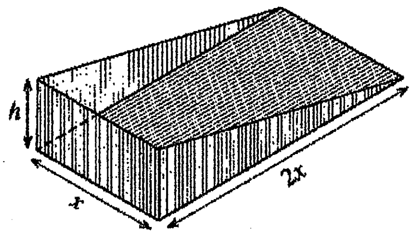
Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x)=\dfrac{9(1-x)}{(x-3)^3}$ for $x\neq 3$.

Hence, find the interval on which $f(x)$ is increasing and the intervals on which $f(x)$ is decreasing. Also find the coordinates of the turning point of $f(x)$.

Sketch the graph of $y=f(x)$ indicating the asymptotes, the turning point and the x-intercepts.

Using the graph, find all real values of x satisfying the inequality $\dfrac{1}{1+f(x)}\leq\dfrac{1}{3}$.

(*b*) The adjoining figure shows the portion of a dust pan without its handle. Its dimensions in centimetres, are shown in the figure. It is given that its volume x^2h cm^3 is 4500 cm^3. Its surface area S cm^2 is given by $S=2x^2+3xh$. Show that S is minimum when $x=15$.



a) For $x\neq 3$; $f(x)=\dfrac{x(2x-3)}{(x-3)^2}$

Then, $f'(x)=\dfrac{1}{(x-3)^2}\left[2x-3+2x\right]-\dfrac{2x(2x-3)}{(x-3)^3}$ (20)

$$=\frac{(x-3)(4x-3)-2x(2x-3)}{(x-3)^3}$$

$$=\frac{4x^2-15x+9-4x^2+6x}{(x-3)^3}$$

$$=\frac{9(1-x)}{(x-3)^3}$$ (5)

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$f'(x)=0\Leftrightarrow x=1.$ (5)

	$-\infty<x<1$	$1<x<3$	$3<x<\infty$
Sign of $f'(x)$	(-)	(+)	(-)
$f(x)$ is	↘ Decreasing	↗ Increasing	↘ Decreasing
	(5)	(5)	(5)

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Turning point : $\left(1, -\frac{1}{4}\right)$ is a local minimum. (5)

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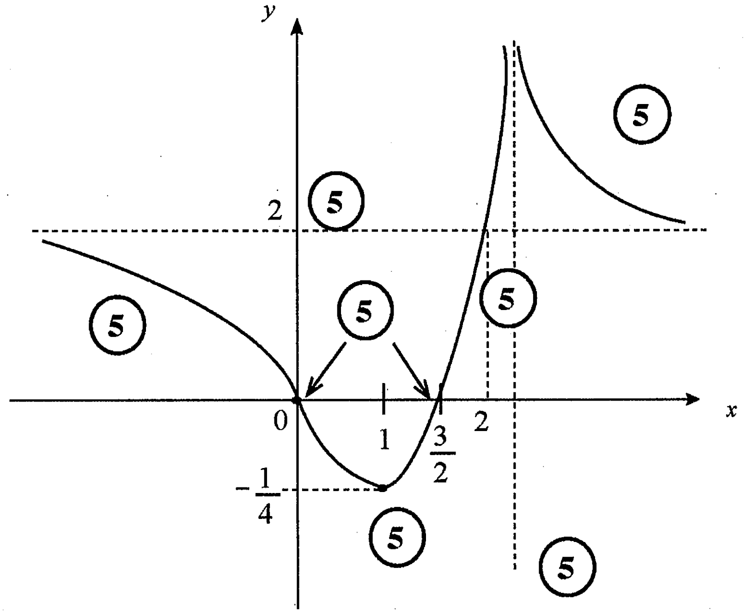
Horizontal asymptote : $\lim_{x \to \pm\infty} f(x) = 2 \quad \therefore\ y = 2$ (5)

Vertical asymptote : $x = 3$. (5)



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$$\frac{1}{1+f(x)} \leq \frac{1}{3}.$$

Note that $1 + f(x) > 0$.

$\therefore 3 \leq 1 + f(x)$.

$\therefore f(x) \geq 2$. (5)

$f(x) = 2 \Leftrightarrow x(2x-3) = 2(x-3)^2$. (5)

$\Leftrightarrow \quad 2x^2 - 3x = 2\{x^2 - 6x + 9)$

$\Leftrightarrow \quad x = 2$ (5)

The required values x are $2 \leq x \leq 3$ or $x > 3$.

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(*b*) $x^2 h = 4500.$

Hence, $S = 2x^2 + 3xh$

$$= 2x^2 + 3x \cdot \frac{4500}{x^2} \quad \text{for } x > 0.$$ (5)

$$\therefore \frac{dS}{dx} = 4x - 3 \times 4500\left(\frac{1}{x^2}\right) = \frac{4(x^3 - 3375)}{x^2}.$$ (5)

$\frac{dS}{dx} = 0$ (5) $\Leftrightarrow x = 15.$ (5)

For $0 < x < 15$, $\frac{ds}{dr} < 0$ (5) and for $x > 15$, $\frac{ds}{dr} > 0.$ (5)

$\therefore S$ is minimum when $x = 15.$ (5)

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