(07) Mathematics

Structure of the Question Paper

Paper I - **Time : 03 hours** (In addition, 10 minutes for reading.)
This paper consists of **two** parts.
Part A : Ten questions. **All** questions should be answered. 25 marks for each question - altogether 250 marks.
Part B : Seven questions. **Five** questions should be answered. Each question carries 150 marks - altogether 750 marks.
Total marks for paper I = 1000

Paper II - **Time : 03 hours** (In addition, 10 minutes for reading.)
This paper consists of **two** parts.
Part A : Ten questions. **All** questions should be answered. 25 marks for each question - altogether 250 marks.
Part B : Seven questions. **Five** questions should be answered. Each question carries 150 marks - altogether 750 marks.
Total marks for paper II = 1000

Calculation of the final mark :
Paper I = 1000
Paper II = 1000
Final mark = 2000 ÷ 20 = 100

(07) Mathematics

Paper I

Part A

1. Let $A = \{x \in \mathbb{R} : \ | x + 3 | < 2 \}$ and $B = \{x \in \mathbb{R} : | x | \geq 4\}$ be subsets of the universal set $\mathbb{R}$. Find $A \cap B$ and $A' \cap B$.

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2. Let A and B be subsets of a universal set S. The set $A \setminus B$ is defined, in the usual notation, by $A \setminus B = A \cap B'$. Show that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ and $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

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3. Let p and q be propositions. Show that the compound propositions $\sim(p \vee (\sim p \wedge q))$ and $\sim p \wedge \sim q$ are logically equivalent.

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4. Using the method of **proof by contradiction**, prove that if $3n^2 + 2$ is odd, then n is odd.

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5. Solve the simultaneous equations $y - \frac{1}{3}\ \log_2 x = 0$ and $8^{2y-1} - 2(x - 4) = 0$ for x and y.

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6. Find all real values of x satisfying the inequality $x - \frac{4}{x} \leq 3$.

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7. Let $f(x) = \sqrt{x+3} - 5$ be a function defined on $[-3,\infty)$. Find the range of the function f and show that f is one-to-one. Find $f^{-1}(x)$.

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8. The line l has gradient -3 and passes through the point $A(2,1)$. A point B is on the line l such that the distance AB is $3\sqrt{10}$. Find the possible coordinates for the point B.

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9. Find the points at which the tangents to the parametric curve given by $x = 2t^3$, $y = 2 - 4t + t^2$ has a slope of -1.

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10. Find the area of the region bounded by the curves $y = x^2$ and $x + y = 2$.

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Part B

11. (*a*) Fifty students sat for an examination in the subjects, Mathematics, Physics and Chemistry. Out of these 50 students, 37 passed Mathematics, 24 passed Physics and 43 passed Chemistry. Further, it is given that at most 19 students passed Mathematics and Physics, at most 29 passed Mathematics and Chemistry and at most 20 passed Physics and Chemistry. Find the largest possible number of students that could have passed all three subjects.

(*b*) Determine whether the compound proposition $[\sim p \wedge (p \vee q)] \rightarrow q$ is a tautology or a contradiction.

12. (*a*) Using the **Principle of Mathematical Induction**, prove that

$$\sum_{r=1}^{n}(3r^2 + 5r + 1) = n(n+2)^2 \text{ for all } n \in \mathbb{Z}^+.$$

(*b*) Let $U_r = \dfrac{2}{(2r-1)(2r+1)}$ for $r \in \mathbb{Z}^+$.

Verify that $U_r = \dfrac{1}{(2r-1)} - \dfrac{1}{(2r+1)}$ for $n \in \mathbb{Z}^+$, and show that $\sum_{r=1}^{n} U_r = \dfrac{2n}{2n+1}$ for $n \in \mathbb{Z}^+$.

Also, find $\sum_{r=10}^{20}(2U_r + 3r)$.

13. (*a*) The roots of the quadratic equation $x^2 + (4+k)x - (25+k) = 0$ are α and $-\alpha^2$, where k is a real constant.

Show that α is a root of the equation $x^3 - x^2 + x - 21 = 0$.

Show that $(x-3)$ is a factor of $x^3 - x^2 + x - 21$ and show that the equation $x^3 - x^2 + x - 21 = 0$ has only one real root.
Hence, find the value of k.

(*b*) Let $f(x) = -2x^2 + 12x - 16$.

Write the function $f(x)$ in the form $a(x-h)^2 + k$, where a, h and k are constants to be determined.

Find the coordinates of the vertex, equation of the axis of symmetry, and the maximum value of f. Sketch the graph of the function $y = f(x)$.

The function g is defined by $g(x) = -2 - f(x+1)$. Determine the axis of symmetry, and the minimum value of the function g.

14. (*a*) Write down, in the usual notation, the binomial expansion of $(a+b)^n$, where a and b real numbers and n is a positive integer.

(i) If the sum of the coefficients of the first, second and the third terms of the binomial expansion of $\left(x^2 + \dfrac{1}{x}\right)^n$ is 46, find n.

(ii) Find the value of k if the coefficient of x^4 in the expansion of $\left(kx + \dfrac{1}{x}\right)^{10}$ is equal to $\dfrac{15}{16}$. For this value of k, find the term of the expansion that is independent of x.

(*b*) A person has the following 3 investment options:

Option 1: Invest under 14% simple interest per annum
Option 2: Invest under 12% compound interest per annum
Option 3: Invest under quarterly compounded 8% interest per annum

(i) Select the best investment option based on the interest accumulated at the end of 5 years.

(ii) The person also has the 4th option of investment where the interest is calculated quarterly at an annual rate of r%. If the interest under option 4 is larger than that is under option 2 for a period of 10 years, what is the minimum value of r?

15. Let $y = m_1 x + c_1$, $y = m_2 x + c_2$ and $x = 0$ be the equations of the sides AB, BC and AC of the triangle ABC respectively. Show that the area of triangle ABC is given by $\dfrac{(c_1 - c_2)^2}{2|m_1 - m_2|}$.

Let $3x - y + 5 = 0$, $2x + 3y - 1 = 0$ and $x + 2y - 3 = 0$ be the equations of the sides BC, CA and AB respectively of the triangle ABC.

A straight line passing through the point A with gradient $-\dfrac{1}{3}$ intersects at the point D with a straight line passing through the point B and parallel to CA. If O is the origin, show that the equation of OD is given by $y + x = 0$.

The straight line passing through the point D and perpendicular to the side AB meet the y -axis at the point E. Find the area of the triangle ODE.

16. (*a*) Find $\lim\limits_{x \to 2} \dfrac{\sqrt{x} - \sqrt{2}}{x^2 - 4}$.

(*b*) Differentiate each of the following with respect to x.

(i) $\left(\dfrac{x}{1 - x}\right)^6$

(ii) $\dfrac{e^{2x} + e^{-2x}}{e^{2x} - e^{-2x}}$

(iii) $x^2 \ln (x^4 + 1)$

(*c*) An open tank of volume $4000\,\text{m}^3$ having a square base and vertical walls is to be constructed from thin sheet material. Find the dimensions of the tank such that the material used is a minimum.

17. (*a*) Using **integration by parts**, evaluate $\displaystyle\int_0^1 x^2 e^{2x}\,\mathrm{d}x$

(*b*) Using **partial fractions**, find $\displaystyle\int \frac{2x + 3}{(x + 1)(x + 2)^2}\,\mathrm{d}x$.

(*c*) The following table gives the values of the function $f(x) = \sqrt{2x + 1}$, correct to three decimal places for values of x between 0 and 1 at intervals of length 0.25.

x	0	0.25	0.50	0.75	1.00
$f(x)$	1	1.225	1.414	1.581	1.732

Using **Simpson's rule**, find an approximate value for $I = \int_0^1 \sqrt{2x + 1}\, dx$ correct to two decimal places.

Using the substitution $u = 2x + 1$, find I and compare the value of I with the approximate value obtained above.

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