Confidential

Department of Examinations – Sri Lanka

G.C.E. (A/L) Examination – 2024

07 – Mathematics I

Marking Scheme

This has been prepared for the use of marking examiners. Some changes would be made according to the views presented at the Chief Examiners' meeting.

Amendments to be included

G. C. E (Advanced Level) Examination – 2024

07 - Mathematics I

Distribution of Marks

Paper I

Part A	= 10×25	=	**250**
Part B	= 05×150	=	**750**

Total = $\dfrac{1000}{10}$

Final marks = **100**

Common Techniques of Marking Answer Scripts.

It is compulsory to adhere to the following standard method in marking answer scripts and entering marks into the mark sheets.

1. Use a red color ball point pen for marking. (Only Chief/Additional Chief Examiner may use a mauve color pen.)
2. Note down Examiner's Code Number and initials on the front page of each answer script.
3. Write off any numerals written wrong with a clear single line and authenticate the alterations with Examiner's initials.
4. Write down marks of each subsection in a △ and write the final marks of each question as a rational number in a □ with the question number. Use the column assigned for Examiners to write down marks.

Example: **Question No. 03**

(i) .... √ △ $\frac{4}{5}$

(ii) .... √ △ $\frac{3}{5}$

(iii) .... √ △ $\frac{3}{5}$

(03) (i) $\frac{4}{5}$ + (ii) $\frac{3}{5}$ + (iii) $\frac{3}{5}$ = □ $\frac{10}{15}$

MCQ answer scripts: (Template)

1. Marking templets for G.C.E.(A/L) and GIT examination will be provided by the Department of Examinations itself. Marking examiners bear the responsibility of using correctly prepared and certified templates.
2. Then, check the answer scripts carefully. If there are more than one or no answers Marked to a certain question write off the options with a line. Sometimes candidates may have erased an option marked previously and selected another option. In such occasions, if the erasure is not clear write off those options too.
3. Place the template on the answer script correctly. Mark the right answers with a 'V' and the wrong answers with a 'X' against the options column. Write down the number of correct answers inside the cage given under each column. Then, add those numbers and write the number of correct answers in the relevant cage.

Structured essay type and assay type answer scripts:

1. Cross off any pages left blank by candidates. Underline wrong or unsuitable answers. Show areas where marks can be offered with check marks.

2. Use the right margin of the overland paper to write down the marks.
3. Write down the marks given for each question against the question number in the relevant cage on the front page in two digits. Selection of questions should be in accordance with the instructions given in the question paper. Mark all answers and transfer the marks to the front page, and write off answers with lower marks if extra questions have been answered against instructions.

4. Add the total carefully and write in the relevant cage on the front page. Turn pages of answer script and add all the marks given for all answers again. Check whether that total tallies with the total marks written on the front page.

Preparation of Mark Sheets.

Except for the subjects with a single question paper, final marks of two papers will not be calculated within the evaluation board this time. Therefore, add separate mark sheets for each of the question paper. Write paper 01 marks in the paper 01 column of the mark sheet and write them in words too. Write paper II Marks in the paper II Column and wright the relevant details.

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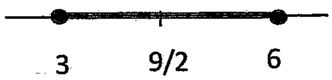
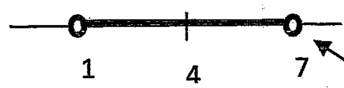
Part A

1. Let $A = \{x \in \mathbb{R} : 5 - |2x-9| \geq 2\}$ and $B = \{x \in \mathbb{R} : |2x-8| < 6\}$. Find $A \cup B$, $A \cap B$ and $B - A$.

$A = \{x \in \mathbb{R}: 5 - |2x - 9| \geq 2\}$ and $B = \{x \in \mathbb{R}: |2x - 8| < 6\}$.

$\therefore x \in A \Leftrightarrow |2x - 9| \leq 3$ and $x \in B \Leftrightarrow |x - 4| < 3.$





(5) or or

$\therefore A = \{x \in \mathbb{R}: 3 \leq x \leq 6\}$ and $B = \{x \in \mathbb{R}: 1 < x < 7\}$

$A = [3,6]$ and $B = (1,7)$

or or (5)

Now,

$A \cup B = \{x \in \mathbb{R}: 1 < x < 7\} = (1,7)$ (5)

$A \cap B = \{x \in \mathbb{R}: 3 \leq x \leq 6\} = [3,6]$ (5)

$B - A = \{x \in \mathbb{R}: 1 < x < 3\} \cup \{x \in \mathbb{R}: 6 < x < 7\} = (1,3) \cup (6,7)$ (5)

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2. Let A, B and C be subsets of a universal set S. Show that $A-(A\cap B\cap C)=A-(B\cap C)$.

$A-(A\cap B\cap C)=A\cap(A\cap B\cap C)'$ (Using the definition) (5)

$=A\cap(A'\cup(B\cap C)')$ (De Morgan's law) (5)

$=(A\cap A')\cup[A\cap(B\cap C)']$ (Distributive property) (5)

$=\emptyset\cup[A\cap(B\cap C)']$ (Empty set) (5)

$=A\cap(B\cap C)'$

$=A-(B\cap C)$ (Using the definition) (5)

25

3. Show that the compound proposition $(p \Rightarrow q) \vee (p \wedge q)$ is logically equivalent to the compound proposition $\sim p \vee q$.

p	q	$p \Rightarrow q$	$p \wedge q$	$(p \Rightarrow q) \vee (p \wedge q)$	$\sim p$	$\sim p \vee q$
T	T	T	T	T	F	T
T	F	F	F	F	F	F
F	T	T	F	T	T	T
F	F	T	F	T	T	T
5		5	5	5		5

The truth values of columns 5 and 7 are identical.

$\therefore (p \Rightarrow q) \vee (p \wedge q)$ and $\sim p \vee q$ are logically equivalent.

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4. Let $n \in \mathbb{Z}$. Using the method of contradiction, prove that if $n^3 + 2n + 7$ is odd, then n is even.

Suppose that $n^3 + 2n + 7$ is odd and n is odd. (5)

Then, $n = 2k + 1$, where $k \in \mathbb{Z}$. (5)

$$n^3 + 2n + 7 = (2k + 1)^3 + 2(2k + 1) + 7.$$

$$= 8k^3 + 12k^2 + 6k + 1 + 4k + 2 + 7.$$ (5)

$$= 2(4k^3 + 6k^2 + 5k + 5).$$ (5)

Since $4k^3 + 6k^2 + 5k + 5 \in \mathbb{Z}$, $n^3 + 2n + 7$ is even.

This is a contradiction. (5)

$\therefore$ if $n^3 + 2n + 7$ is odd, then n is even.

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5. Obtain a quadratic equation in x from the equation $\log_{10}(x+15) = 2 - \log_{10} x$. **Hence,** find the value of x satisfying the equation $\log_{10}(x+15) = 2 - \log_{10} x$.

$\log_{10}(x+15) = 2 - \log_{10} x$

$\Rightarrow \log_{10}(x+15) + \log_{10} x = 2$

$\Rightarrow \log_{10} x(x+15) = 2$ (5)

$\Rightarrow x(x+15) = 10^2$ (5)

$\Rightarrow x^2 + 15x - 100 = 0$ (5)

$\Rightarrow (x-5)(x+20) = 0$

$\Rightarrow x = 5$ or $x = -20$ (5)

$x = -20$ does not satisfies the equation.

$x = 5$ satisfy the equation.

$\therefore x = 5.$ (5)

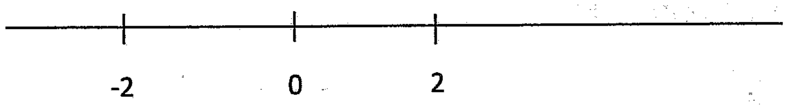
25

6. Find all real values of x that satisfy the inequality $\frac{x}{x-2}-\frac{1}{x}<1$.

$$\frac{x}{x-2}-\frac{1}{x} < 1$$

$$\Leftrightarrow \frac{x^2-(x-2)-x(x-2)}{x(x-2)} < 0$$

$$\Leftrightarrow \frac{x+2}{x(x-2)} < 0$$ (5)



For identifying -2, 0, and 2 (5)

	$x<-2$	$-2<x<0$	$0<x<2$	$2<x$
Sign of $\frac{x+2}{x(x-2)}$	$\frac{(-)}{(-)(-)} = (-)$	$\frac{(+)}{(-)(-)} = (+)$	$\frac{(+)}{(+)(-)} = (-)$	$\frac{(+)}{(+)(+)} = (+)$

All 4 (10)

Any 2 or 3 (5)

∴The required answer: $x<-2$ or $0<x<2$ (5)

: $(-\infty,-2)\cup(0,2)$

25

7. The functions f and g are defined by $f(x) = 3x + 2$ and $g(x) = \frac{6}{2x+3}$.
Find the value of x for which $f(g(x)) = 3$.
Solve the equation $f^{-1}(g(x)) = 1$ for x.

$f(g(x)) = 3g(x) + 2$ (5)

$= 3 \cdot \frac{6}{(2x+3)} + 2$

$= \frac{18 + 4x + 6}{2x + 3}$ (5)

$f(g(x)) = 3$

$\Leftrightarrow 4x + 24 = 3(2x + 3)$

$\Leftrightarrow 2x = 15$

$\Leftrightarrow x = \frac{15}{2}$ (5)

Interchanging x and y in $y = 3x + 2$, we obtain

$x = 3y + 2$

$\therefore y = \frac{x - 2}{3}$

$\therefore f^{-1}(x) = \frac{x - 2}{3}$

$f^{-1}(g(x)) = \frac{g(x) - 2}{3} = \frac{\frac{6}{(2x+3)} - 2}{3} = \frac{6 - 4x - 6}{3(2x + 3)}$ (5)

$f^{-1}(g(x)) = 1 \Leftrightarrow -4x = 6x + 9$

$\Leftrightarrow x = -\frac{9}{10}$ (5)

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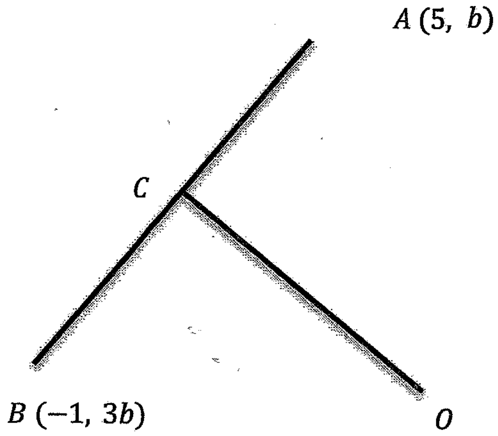
8. Let $A \equiv (5, b)$ and $B \equiv (-1, 3b)$ be two points, where $b \in \mathbb{R}$. Also, let C be the mid-point of AB. Find the values of b such that AB is perpendicular to OC, where O is the origin.



$C \equiv (2\,,\ 2b)$ (5)

The gradient of $AB = m_1 = \frac{2b}{-6}$ (5)

The gradient of $OC = m_2 = \frac{2b}{2}$ (5)

$AB \perp OC \Leftrightarrow m_1 \cdot m_2 = -1$ (5)

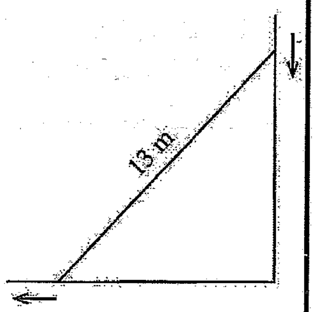
$\Leftrightarrow \frac{-2b}{6} \times \frac{2b}{2} = -1$

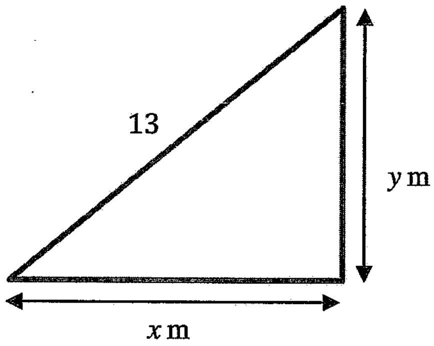
$\Leftrightarrow b^2 = 3$

$\Leftrightarrow b = \pm\sqrt{3}$ (5)

25

9. A 13 m long ladder is resting on a horizontal ground, while leaning against a vertical wall (see the figure). The foot of the ladder is pulled away from the wall at a rate of 0.2 m s^{-1}. Find the rate at which the top of the ladder is sliding down the wall when the foot of the ladder is at a distance 12 m from the wall.





$\frac{dx}{dt} = 0.2\, ms^{-1}$

$$x^2 + y^2 = 169 \quad \text{———} \quad (1)$$ (5)

$\frac{d(1)}{dt}$: $$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \quad \text{———} \quad (2)$$ (5)

When $x = 12$, $(1) \Rightarrow 144 + y^2 = 169$

$$\therefore y = 5 \quad (\because y > 0)$$ (5)

Now,

$(2) \Rightarrow 12 \times 0.2 + 5\left.\frac{dy}{dt}\right|_{x=12} = 0$ (5)

$\therefore \left.\frac{dy}{dt}\right|_{x=12} = -\frac{12\times0.2}{5} = -\frac{12}{25}\, ms^{-1}$ (5)

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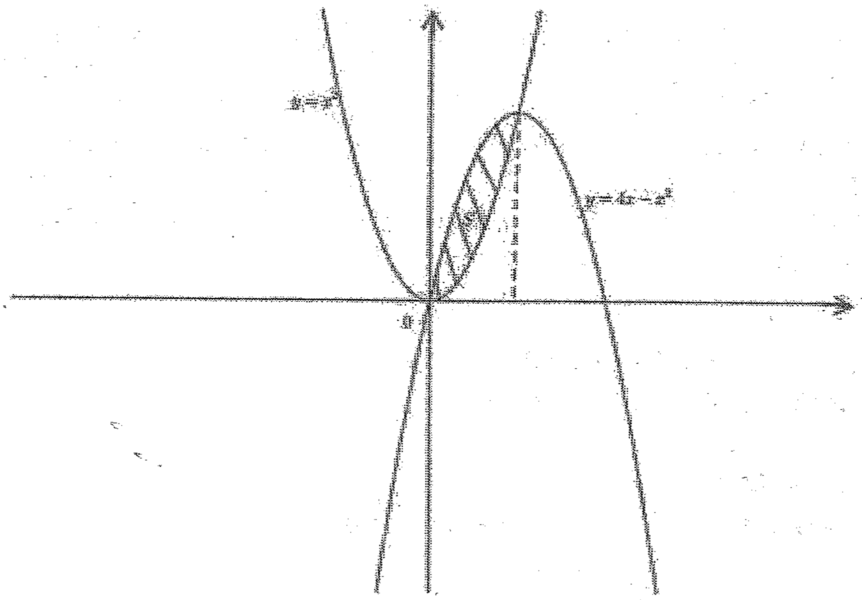
10. Find the area of the region enclosed by the curves $y=x^2$ and $y=4x-x^2$.



At the points of intersection, $x^2 = 4x - x^2$.

This gives us $x = 0$ or $x = 2$. (5)

The required area $= \int_0^2 \{(4x - x^2) - x^2\}\, dx$ (10)

$$= \int_0^2 (4x - 2x^2)\, dx$$

$$= \left(\frac{4x^2}{2} - \frac{2x^3}{3}\right)\Big|_0^2$$ (5)

$$= 8 - \frac{16}{3} = \frac{8}{3}$$ (5)

25

Part B

* *Answer five questions only.*

11. (*a*) While observing the G.C.E. (Ordinary Level) results for the subjects Mathematics, Science and English of 40 students in a G.C.E. (Advanced Level) class, the following information were revealed:

- 20 students for Mathematics and 18 students for Science have obtained A grades.
- 9 students for both Mathematics and Science, 8 students for both Science and English, and 5 students for both Mathematics and English have obtained A grades.
- 10 students have obtained A grades only for two of the subjects among these subjects.
- The number of students who have not obtained A grades for any of these three subjects is twice the number of students who have obtained A grades for all three subjects.

Find the number of students

(i) who have obtained A grades for all three subjects,

(ii) who have obtained A grade only for Mathematics,

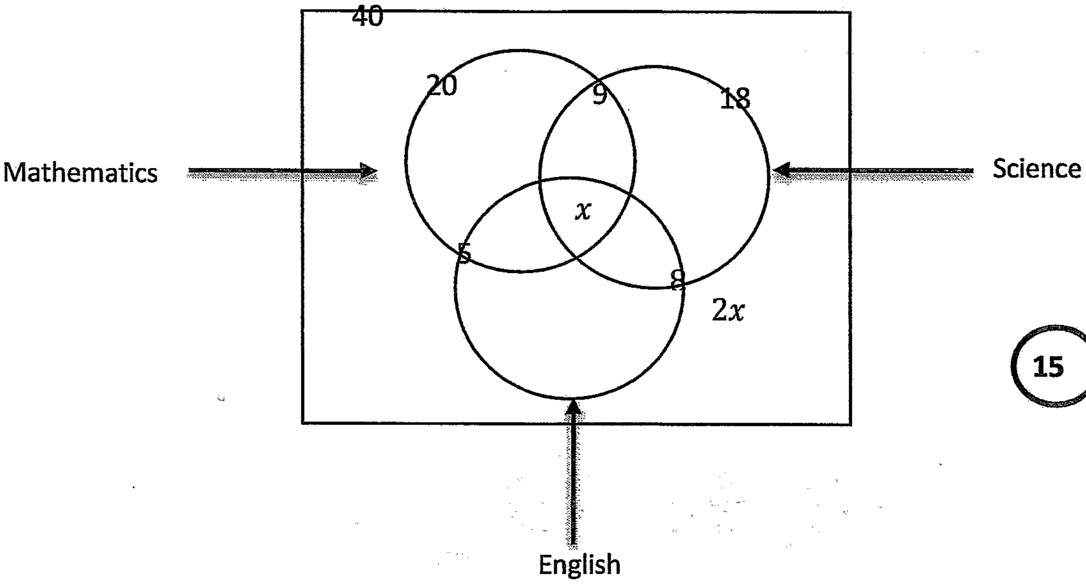
(iii) who have obtained A grade for English.

(*b*) Using truth tables, show that

(i) $(p \wedge q) \vee r$ is logically equivalent to $(p \vee r) \wedge (q \vee r)$,

(ii) $(p \vee q) \wedge (\sim p \vee r) \Rightarrow q \vee r$ is a tautology.

a)



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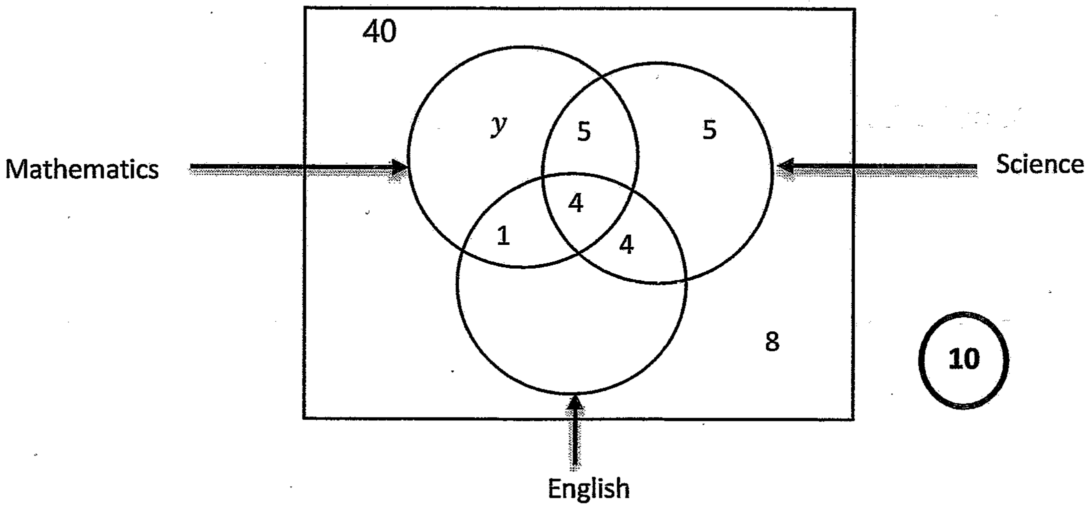
(i) Let x be the number of students who have obtained A grades for all 3 subjects.

Given: $(9 - x) + (5 - x) + (8 - x) = 10$ 20

$22 - 3x = 10$

$\therefore x = 4.$ 5



(10)

(ii) Let y be the number of students who have obtained A grade only for Mathematics.

Then $20 = y + 9 + 1$ (10)

$\therefore y = 10$ (5)

(iii) The number of students who have obtained A grade for English

$= 40 - (10 + 5 + 5 + 8)$ (10)

$= 12$ (5)

80

(b)

(i).

p	q	r	$p \wedge q$	$(p \wedge q) \vee r$	$p \vee r$	$q \vee r$	$(p \vee r) \wedge (q \vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	F	T	T	T	T
F	T	T	F	T	T	T	T
T	F	F	F	F	T	F	F
F	T	F	F	F	F	T	F
F	F	T	F	T	T	T	T
F	F	F	F	F	F	F	F

(10) (10) (10)

The truth values of the columns 5 and 8 are identical. (5)

Hence the result.

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(ii).

p	q	r	$p \vee q$	$\sim p$	$\sim p \vee r$	(a) $(p \vee q) \wedge (\sim p \vee r)$	(b) $q \vee r$	$(a) \Rightarrow (b)$
T	T	T	T	F	T	T	T	T
T	T	F	T	F	F	F	T	T
T	F	T	T	F	T	T	T	T
F	T	T	T	T	T	T	T	T
T	F	F	T	F	F	F	F	T
F	T	F	T	T	T	T	T	T
F	F	T	F	T	T	F	T	T
F	F	F	F	T	T	F	F	T

(10) (5) (10) (5) (5)

35

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12. (a) Using the **Principle of Mathematical Induction**, prove that

$$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{(2n+1)} \text{ for all } n \in \mathbb{Z}^+.$$

(b) Let $f(r) = \frac{r}{(r+2)^2}$ for $r \in \mathbb{Z}^+$.

Find U_r such that $U_r = f(r) - f(r+1)$ for $r \in \mathbb{Z}^+$.

Show that $\sum_{r=1}^{n} U_r = \frac{1}{9} - \frac{(n+1)}{(n+3)^2}$ for $n \in \mathbb{Z}^+$.

Hence, show that $\sum_{r=1}^{\infty} U_r$ is convergent and find its sum.

Also, find $\sum_{r=15}^{\infty} U_r$.

(a) For $n = 1$,

L.H.S. $= \frac{1}{3}$ (5) and R.H.S. $= \frac{1}{3}$ (5)

$\therefore$ The result is true for $n = 1$. (5)

Take any $p \in \mathbb{Z}^+$ and assume that the result is true for $n = p$. (5)

$$i.e. \sum_{r=1}^{p} \frac{1}{(2r-1)(2r+1)} = \frac{p}{2p+1}$$ (5)

$$\text{Now, } \sum_{r=1}^{p+1} \frac{1}{(2r-1)(2r+1)} = \sum_{r=1}^{p} \frac{1}{(2r-1)(2r+1)} + \frac{1}{(2p+1)(2p+3)}$$ (5)

$$= \frac{p}{(2p+1)} + \frac{1}{(2p+1)(2p+3)}$$ (5)

$$= \frac{p(2p+3)+1}{(2p+1)(2p+3)}$$

$$= \frac{2p^2+3p+1}{(2p+1)(2p+3)}$$ (5)

$$= \frac{(2p+1)(p+1)}{(2p+1)(2p+3)}$$

$$= \frac{p+1}{2p+3} \quad (5)$$

Hence, if the result is true for $n = p$, then it is true for $n = p + 1$. (5)

We have already proved that the result is true for $n = 1$.

Hence, by the Principle of Mathematical Induction, the result is true for all $n \in \mathbb{Z}^+$.

(5) **55**

(b)

$$U_r = f(r) - f(r+1)$$

$$= \frac{r}{(r+2)^2} - \frac{r+1}{(r+3)^2} \quad (10)$$

$$= \frac{r(r^2+6r+9) - (r+1)(r^2+4r+4)}{(r+2)^2(r+3)^2} \quad (5)$$

$$= \frac{r^3+6r^2+9r-r^3-4r^2-4r-r^2-4r-4}{(r+2)^2(r+3)^2} \quad (5)$$

$$= \frac{r^2+r-4}{(r+2)^2(r+3)^2} \quad (10)$$

30

$r = 1$: $u_1 = f(1) - f(2)$

$r = 2$: $u_2 = f(2) - f(3)$ (10)

$\vdots$

$r = n-1$: $u_{n-1} = f(n-1) - f(n)$

$r = n$: $u_n = f(n) - f(n+1)$ (10)

$$\sum_{r=1}^{n} u_r = f(1) - f(n+1) \quad (10)$$

$$\therefore \sum_{r=1}^{n} u_r = \frac{1}{9} - \frac{1}{(n+3)^2}$$ (5)

$$\therefore \lim_{n\to\infty} \sum_{r=1}^{n} u_r = \lim_{n\to\infty} \left\{\frac{1}{9} - \frac{1}{(n+3)^2}\right\}$$

$$= \frac{1}{9}.$$ (5)

$\therefore \sum_{r=1}^{\infty} u_r$ is convergent and has sum $= 1/9$.

(5) (5)

50

$$\sum_{r=15}^{\infty} u_r = \sum_{r=1}^{\infty} u_r - \sum_{r=1}^{14} u_r$$ (10)

$$= \frac{1}{9} - \left(\frac{1}{9} - \frac{15}{17^2}\right)$$

$$= \frac{15}{289}.$$ (5)

15

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13. (*a*) The roots of the equation $x^2 + px + q = 0$ are -3 and 5; where $p, q \in \mathbb{R}$.
Find the values of p and q.
Using these values of p and q, find the value of the constant $r (\in \mathbb{R})$ for which the equation, $x^2 + px + q + r = 0$ has equal roots.

(*b*) Let $p(x) = 2x^3 + ax^2 + (a+4)x + 6$, where $a \in \mathbb{R}$. It is given that $(x-2)$ is a factor of $p(x)$.
Find the value of a.
When a has this value, factorize $p(x)$.
Also, solve the inequality $p(x) > 0$.

a) Substitute $x = -3$: $9 - 3p + q = 0$ .... (1) (10)

Substitute $x = 5$: $25 + 5q + q = 0$ .... (2) (10)

(1) and (2) $\Rightarrow$ $p = -2$ (5) and $q = -15$ (5)

30

$x^2 - 2x - 15 + r = 0$

$\Delta = 4 - 4(-15 + r) = 0$ (15)

$\therefore \; 4 + 60 - 4r = 0$

$\therefore \; r = 16$ (5)

20

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b) $(x-2)$ is a factor of $p(x) = 2x^3 + ax^2 + (a+4)x + 6$

$\Leftrightarrow p(2) = 0$ (10) (10)

$\Leftrightarrow 16 + 4a + 2(a+4) + 6 = 0$

$\Leftrightarrow 6a + 30 = 0$

$\Leftrightarrow a = -5$ (5)

25

$$p(x) = 2x^3 - 5x^2 - x + 6$$
$$= (x-2)(2x^2 - x - 3) \quad (10)$$
$$= (x-2)(2x-3)(x+1) \quad (10)$$

20

$$p(x) = 0 \Leftrightarrow x = 2 \text{ or } x = \frac{3}{2} \text{ or } x = -1 \quad (5)$$

	$-\infty < x < -1$	$-1 < x < \frac{3}{2}$	$\frac{3}{2} < x < 2$	$2 < x < \infty$
Sign of $p(x)$	$(-)(-)(-) = (-)$	$(-)(-)(+) = (+)$	$(-)(+)(+) = (-)$	$(+)(+)(+) = (+)$

(40)

$$p(x) > 0 \Leftrightarrow -1 < x < \frac{3}{2} \text{ or } 2 < x < \infty$$
$$\Leftrightarrow x \in \left(-1, \frac{3}{2}\right) \cup (2, \infty) \quad (10)$$

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14.(*a*) Find the first 3 terms in the expansion of $(1-x-x^2)^6$ in ascending powers of x.

Hence, find the value of k such that the coefficient of x^2 in the expansion of $(k+x^2)(1-x-x^2)^6$ is 10.

(*b*) A person takes a loan of Rs. 500 000 from a bank that charges a monthly interest of 1%. He pays back the loan in equal monthly instalments of Rs. B over 5 years. Let Rs. A_n be the amount to be paid back after n months. Show that

$$A_2 = 500\,000\ (1.01)^2 - B(1+1.01) \text{ and}$$

$$A_3 = 500\,000\ (1.01)^3 - B(1+1.01+1.01^2).$$

Write down a similar expression for A_n in terms of $n(\leq 60)$ and B.

Find the value of B.

a) $(1-x-x^2)^6 = [1-(x+x^2)]^6$

$$= {}^6c_0 - {}^6c_1(x+x^2) + {}^6c_2(x+x^2)^2 + \cdots \quad (10)$$

$$= 1 - 6(x+x^2) + 15(x^2+2x^3+x^4) + \cdots \quad (10)$$

$$= 1 - 6x + 9x^2 + \cdots \quad (10)$$

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$(k+x^2)(1-x-x^2)^6 = (k+x^2)(1-6x+9x^2+\cdots)$

The coefficients of x^2 in the expansion of this $= 9k+1$ (15)

Given : $9k+1 = 10$ (5)

$\therefore k = 1$ (5)

25

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b)

$A_1 = 500\,000 + 500\,000 \times \frac{1}{100} - B$ (10)

$= 500\,000\,(1.01) - B$ (5)

$A_2 = A_1(1.01) - B$ (5)

$= [\,500\,000\,(1.01) - B\,](1.01) - B$ (5)

$= 500\,000\,(1.01)^2 - B(1 + 1.01)$ (5)

$A_3 = A_2\,(1.01) - B$ (5)

$= [\,500\,000\,(1.01)^2 - B(1 + 1.01)](1.01) - B$ (5)

$= 500\,000\,(1.01)^3 - B(1 + 1.01 + 1.01^2)$ (5)

45

$A_n = 500\,000\,(1.01)^n - B(1 + 1.01 + 1.01^2 + \cdots + 1.01^{n-1})$ (15)

15

$A_{60} = 0$ (10)

$\Leftrightarrow \quad 500\,000\,(1.01)^{60} - B(1 + 1.01 + 1.01^2 + \cdots + 1.01^{59}) = 0$ (5)

$\Leftrightarrow \quad 500\,000\,(1.01)^{60} = B\left(\frac{1.01^{60}-1}{1.01-1}\right)$ (10)

$\Leftrightarrow \quad B = \frac{500\,000\,(1.01)^{60}}{100\,((1.01)^{60}-1)}$ (5)

$\approx 11\,122.22$

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15. Let the line l_1 be given by $4y = 3x + 1$. The point $A \equiv (a, 4)$ lies on l_1. Find the value of a.

Suppose that the line l_2 is parallel to l_1 and passing through the point $(4, -3)$. Find the equation of l_2.

Let B be the point of intersection of l_2 and the y-axis, and C be the point on l_1 such that BC is perpendicular to l_1. Find the coordinates of C.

Let D be the point such that $ABCD$ is a parallelogram. Find the coordinates of D.

Also, find the area of $ABCD$.

Since A $\equiv (a, 4)$ lies on l_1,

$4 \times 4 = 3a + 1.$ (10)

$\therefore\ a = 5.$ (5)

15

The gradiant of $l_1 = \frac{3}{4}$. (5)

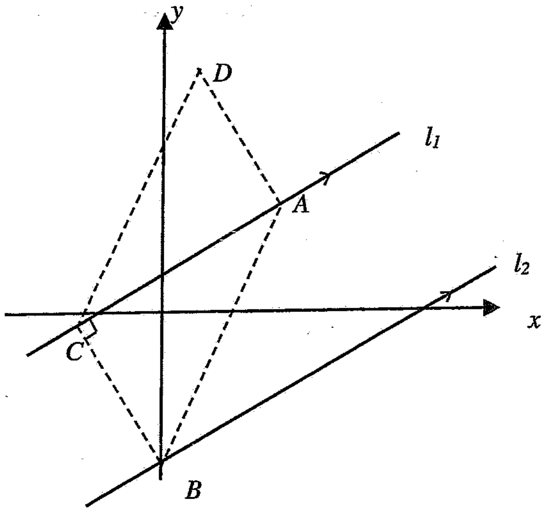
Since l_2 is parallel to l_1, the gradiant of $l_2 = \frac{3}{4}$. (5)

The equation of l_2 is given by

$y + 3 = \frac{3}{4}(x - 4).$ (10)

i.e. $4y = 3x - 24.$ (5)

25



Let $C \equiv (x_0, y_0)$. (10)

Since C is on l_1, $4y_0 = 3x_0 + 1$. --------------- (1)

Substitute $x = 0$ in the equation of l_2.

We get $4y = -24$.

$\therefore\ y = -6$.

$\therefore\ B \equiv (0, -6)$ (10)

The gradiant of BC $= \frac{y_0+6}{x_0}$. (10)

Since BC is perpendicular to l_1, we have

$$\left(\frac{y_0+6}{x_0}\right) \times \frac{3}{4} = -1.$$ (10)

$4x_0 + 3y_0 = -18$ (5) --------------- (2)

$(1) \Rightarrow\ y_0 = \frac{1}{4}(3x_0 + 1)$.

Now $(2) \Rightarrow\ 4x_0 + \frac{3}{4}(3x_0 + 1) = -18$.

$\therefore\ 25x_0 = -75$.

$x_0 = -3$ and

$y_0 = \frac{1}{4}(-9 + 1) = -2$

(15) $\therefore\ C \equiv (-3, -2)$.

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Let D $\equiv (x_1, y_1)$.

The mid point of AC $\equiv \left(\frac{5-3}{2}, \frac{4-2}{2}\right) \equiv (1,1)$. (10)

The mid point of BD $\equiv \left(\frac{x_1}{2}, \frac{y_1-6}{2}\right)$. (10)

Since ABCD is a parallelogram, these two points are the same.

$\therefore \frac{x_1}{2} = 1$ and $\frac{y_1 - 6}{2} = 1$.

$\therefore x_1 = 2$ and $y_1 = 8$.

$\therefore$ D $\equiv (2,8)$. (10)

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$AC = \sqrt{(5+3)^2 + (4+2)^2} = 10.$ (5)

$BC = \sqrt{(-3)^2 + (-2+6)^2} = 5.$ (5)

$\therefore$ The area of Δ ABC $= \frac{1}{2} \times 10 \times 5 = 25.$ (5)

Now, the area of ABCD $= 2 \times$ The area of Δ ABC
$= 50$ (5)

20

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16. (*a*) Evaluate $\lim_{x\to 1} \frac{(x^3-1)^2}{(\sqrt{x}-1)(2x^2-x-1)}$.

(*b*) Differentiate each of the following with respect to x:

(i) $\sqrt{\ln(x^4e^x+5x^2+3)}$, (ii) $(2-3x^2)^7(x+2x^2)^5$, (iii) $\frac{3e^{x^2}+2x^3}{3e^{x^2}-2x^3}$.

(*c*) A right circular hollow cylinder, open at one end, is constructed from a thin sheet of metal. The total outer surface area of the cylinder is 192π cm^2. The cylinder has a radius of r cm and a height of h cm.

Express h in terms of r, and show that the volume V cm^3, of the cylinder is given by $V=\frac{1}{2}\pi(192r-r^3)$.

Find the value of r such that V is maximum.

a)

$$\lim_{x\to 1} \frac{(x^3-1)^2}{(\sqrt{x}-1)(2x^2-x-1)}$$

$$= \lim_{x\to 1} \frac{(x-1)^2(x^2+x+1)^2}{(\sqrt{x}-1)(2x+1)(x-1)}$$ (10)

$$= \lim_{x\to 1} \frac{(x-1)(x^2+x+1)^2}{(\sqrt{x}-1)(2x+1)}$$ (5)

$$= \lim_{x\to 1} \frac{(\sqrt{x}-1)(\sqrt{x}+1)(x^2+x+1)^2}{(\sqrt{x}-1)(2x+1)}$$ (10)

$$= \lim_{x\to 1} \frac{(\sqrt{x}+1)(x^2+x+1)^2}{(2x+1)}$$ (5)

$$= \frac{2\times 3^2}{3}$$

$$= 6$$ (5)

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b)

i) $\frac{d}{dx}\sqrt{\ln(x^4e^x + 5x^2 + 3)}$

$$= \frac{1}{2\sqrt{\ln(x^4e^x + 5x^2 + 3)}} \cdot \frac{1}{(x^4e^x + 5x^2 + 3)} \cdot (x^4e^x + 4x^3e^x + 10x)$$ 15

ii) $\frac{d}{dx}[(2 - 3x^2)^7(x + 2x^2)^5]$

$$= (2 - 3x^2)^7 .\ 5(x + 2x^2)^4(1 + 4x) + (x + 2x^2)^5 .\ 7(2 - 3x^2)^6 .(-6x)$$ 15

iii) $\frac{d}{dx}\left(\frac{3e^{x^2} + 2x^3}{3e^{x^2} - 2x^3}\right)$

$$= \frac{(3e^{x^2} - 2x^3)(6xe^{x^2} + 6x^2) - (3e^{x^2} + 2x^3)(6xe^{x^2} - 6x^2)}{(3e^{x^2} - 2x^3)^2}$$ 20

50

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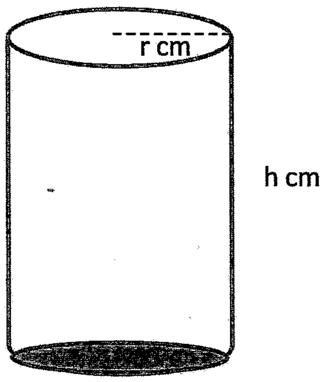
c)



Given: $2\pi rh + \pi r^2 = 192\pi$ 10

$$2rh = 192 - r^2$$ 5

$$\therefore h = \frac{192 - r^2}{2r}$$ 5

Now, $V = \pi r^2 h$ (5)

$= \pi r^2 . \frac{(192-r^2)}{2r}$ (5)

$= \frac{1}{2}\pi(192r - r^3)$ (5)

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$\frac{dV}{dr} = \frac{1}{2}\pi(192 - 3r^2)$ (5)

(5) $\frac{dV}{dr} = 0 \quad \Leftrightarrow \quad 192 - 3r^2 = 0$ (5)

$\Leftrightarrow \quad r^2 = \frac{192}{3} = 64$ (5)

$\Leftrightarrow \quad r = 8 \quad (\because r > 0)$ (5)

$\frac{dV}{dr} > 0$ for $0 < r < 8$ and $\frac{dV}{dr} < 0$ for $r > 8$.

$\therefore V$ is maximum when $r = 8$. (5)

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17.(*a*) Using **partial fractions**, find the value of $\int_1^2 \frac{1}{x^2(x+1)}dx$.

(*b*) Using **integration by parts**, find the value of $\int_1^2 (12x^3+4x)\ln x\, dx$.

(*c*) The following table gives the values of the function $f(x)=\ln(4+x^3)$, correct to three decimal places, for values of x between 1 and 2 at intervals of length 0.25:

x	1	1.25	1.5	1.75	2.0
$f(x)$	1.609	1.784	1.998	2.236	2.485

Using **Simpson Rule**, find an approximate value for $\int_1^2 \ln(4+x^3)\,dx$.

Hence, find an approximate value for $\int_1^2 \ln\left(\frac{1}{4+x^3}\right)dx$.

a) $\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$

$\Rightarrow \; 1 = Ax(x+1) + B(x+1) + Cx^2$

$x = 0;\; B = 1$

$x = -1;\; C = 1$

$x = 1;\, 1 = 2A + 2B + C$

$A = -1$

$\frac{1}{x^2(x+1)} = -\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}$ (20)

$\therefore \int_1^2 \frac{1}{x^2(x+1)}dx = \int_1^2 \left\{-\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\right\}dx$ (10)

$= \left(-\ln|x| - \frac{1}{x} + \ln|x+1|\right)\Big|_1^2$ (20)

$= \left(-\ln 2 - \frac{1}{2} + \ln 3\right) - (-\ln(1) - 1 + \ln 2)$ (5)

$= \ln\left(\frac{3}{4}\right) + \frac{1}{2}$ (5)

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b) $\int_1^2 (12x^3 + 4x)\ln x\,dx$

$= \ln x \cdot (3x^4 + 2x^2)|_1^2 - \int_1^2 (3x^4 + 2x^2)\cdot \frac{1}{x}dx$ (15)

$= 56\ln 2 - \int_1^2 (3x^3 + 2x)dx$ (10)

$u = \ln x,\ dv = (12x^3 + 4x)dx$ $du = \frac{1}{x}dx, v = 3x^4 + 2x^2$

$= 56\ln 2 - \left(3\frac{x^4}{4} + 2\cdot\frac{x^2}{2}\right)\Big|_1^2$ (10)

$= 56\ln 2 - \left(16 - \frac{7}{4}\right) = 56\ln 2 - \frac{57}{4}$ (5)

[40]

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c) $h = 0.25$ (5)

$\int_1^2 \ln(4 + x^3)dx$

$\approx \frac{h}{3}[f(x_0) + 4(f(x_1) + f(x_3)) + 2f(x_2) + f(x_4)]$ (20)

$= \frac{0.25}{3}[1.609 + 4(1.784 + 2.236) + 2(1.998) + 2.485]$ (15)

$= \frac{0.25}{3}[1.609 + 4(4.02) + 2(1.998) + 2.485]$

$= \frac{0.25}{3}[1.609 + 16.08 + 3.996 + 2.485]$ (5)

$= \frac{0.25}{3} \times 24.17$

$= 2.014.$ (5)

[50]

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Department of Examinations – Sri Lanka

G.C.E. (A/L) Examination – 2024

07 – Mathematics II

Marking Scheme

This has been prepared for the use of marking examiners. Some changes would be made according to the views presented at the Chief Examiners' meeting.

Amendments to be included

G. C. E (Advanced Level) Examination – 2024

07 - Mathematics II

Distribution of Marks

Paper I

Part A	=	**10 × 25**	=	**250**
Part B	=	**05 × 150**	=	**750**

Total $= \dfrac{1000}{10}$

Final marks = **100**

Common Techniques of Marking Answer Scripts.

It is compulsory to adhere to the following standard method in marking answer scripts and entering marks into the mark sheets.

1. Use a red color ball point pen for marking. (Only Chief/Additional Chief Examiner may use a mauve color pen.)
2. Note down Examiner's Code Number and initials on the front page of each answer script.
3. Write off any numerals written wrong with a clear single line and authenticate the alterations with Examiner's initials.
4. Write down marks of each subsection in a △ and write the final marks of each question as a rational number in a □ with the question number. Use the column assigned for Examiners to write down marks.

Example: **Question No. 03**

(i)	....	√	△ $\frac{4}{5}$
(ii)	....	√	△ $\frac{3}{5}$
(iii)	....	√	△ $\frac{3}{5}$

(03) (i) $\frac{4}{5}$ + (ii) $\frac{3}{5}$ + (iii) $\frac{3}{5}$ = $\boxed{\frac{10}{15}}$

MCQ answer scripts: (Template)

1. Marking templets for G.C.E.(A/L) and GIT examination will be provided by the Department of Examinations itself. Marking examiners bear the responsibility of using correctly prepared and certified templates.
2. Then, check the answer scripts carefully. If there are more than one or no answers Marked to a certain question write off the options with a line. Sometimes candidates may have erased an option marked previously and selected another option. In such occasions, if the erasure is not clear write off those options too.
3. Place the template on the answer script correctly. Mark the right answers with a 'V' and the wrong answers with a 'X' against the options column. Write down the number of correct answers inside the cage given under each column. Then, add those numbers and write the number of correct answers in the relevant cage.

Structured essay type and assay type answer scripts:

1. Cross off any pages left blank by candidates. Underline wrong or unsuitable answers. Show areas where marks can be offered with check marks.

2. Use the right margin of the overland paper to write down the marks.
3. Write down the marks given for each question against the question number in the relevant cage on the front page in two digits. Selection of questions should be in accordance with the instructions given in the question paper. Mark all answers and transfer the marks to the front page, and write off answers with lower marks if extra questions have been answered against instructions.

4. Add the total carefully and write in the relevant cage on the front page. Turn pages of answer script and add all the marks given for all answers again. Check whether that total tallies with the total marks written on the front page.

Preparation of Mark Sheets.

Except for the subjects with a single question paper, final marks of two papers will not be calculated within the evaluation board this time. Therefore, add separate mark sheets for each of the question paper. Write paper 01 marks in the paper 01 column of the mark sheet and write them in words too. Write paper II Marks in the paper II Column and wright the relevant details.

***

Part A

1. Let $a \in \mathbb{R}$.

Show that $\begin{vmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix} = (1-a^3)^2$.

$$\begin{vmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix} = \begin{vmatrix} 1-a^2 & a-1 & a^2-a \\ a^2-a & 1-a^2 & a-1 \\ a & a^2 & 1 \end{vmatrix} \quad (05)$$

$(-1)R_2 + R_1$

$(-1)R_3 + R_2$

$$= (1-a)^2 \begin{vmatrix} 1+a & -1 & -a \\ -a & 1+a & -1 \\ a & a^2 & 1 \end{vmatrix} \quad (05)$$

$$= (1-a)^2 \begin{vmatrix} 1+a & -1 & -a \\ -a & 1+a & -1 \\ 0 & 1+a+a^2 & 0 \end{vmatrix} \quad (05)$$

$R_2 + R_3$

$$= (1-a)^2(1+a+a^2)(1+a+a^2) \quad (05)$$

$$= (1-a^3)^2 \quad (05)$$

25

2. Let $\mathbf{A}=\begin{pmatrix}3 & -2\\ 4 & -2\end{pmatrix}$. Find $\mathbf{A}^2$.

Find the value of $k(\in \mathbb{R})$ such that $\mathbf{A}^2 = k\mathbf{A} - 2\mathbf{I}$, where $\mathbf{I}$ is the 2×2 identity matrix.

$$A^2 = \begin{pmatrix}3 & -2\\ 4 & -2\end{pmatrix}\begin{pmatrix}3 & -2\\ 4 & -2\end{pmatrix} = \begin{pmatrix}1 & -2\\ 4 & -4\end{pmatrix}$$ (10)

$$A^2 = kA - 2I$$

$$\Leftrightarrow \begin{pmatrix}1 & -2\\ 4 & -4\end{pmatrix} = \begin{pmatrix}3k & -2k\\ 4k & -2k\end{pmatrix} - \begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}$$ (05)

$$\Leftrightarrow \begin{pmatrix}1 & -2\\ 4 & -4\end{pmatrix} = \begin{pmatrix}3k-2 & -2k\\ 4k & -2k-2\end{pmatrix}$$ (05)

$$\Leftrightarrow 1 = 3k-2, -2 = -2k, 4 = 4k, \text{and } -4 = -2k-2$$

$\Leftrightarrow k = 1.$ (05)

25

3. The median of the six data 2, 4, x, 7, 10, y arranged in ascending order is 6. It is given that their range is twice the inter-quartile range. Find the values of x and y.
Also, find the mean deviation of the above data.

$$\frac{x+7}{2} = 6$$

$\therefore x = 5$ (05)

The range $= y - 2$

First quartile $= 4$

Third quartile $= 10$

$\therefore$ IQR $= 10 - 4 = 6$ (05)

Given: $y - 2 = 2\times 6$

$\therefore y = 14$ (05)

The mean $= \frac{2+4+5+7+10+14}{6} = \frac{42}{6} = 7$ (05)

The mean deviation $= \frac{|2-7|+|4-7|+|5-7|+|10-7|+|14-7|}{6}$

$= \frac{5+3+2+3+7}{6} = \frac{10}{3}$ (05)

25

> **4.** The mean of 5 observations is 12 and their variance is 2. When another observation is included the new mean of the 6 observations is 11.
> Find the value of this 6^{th} observation included.
> **Hence,** find the new variance.

Let $x_1, x_2, \dots, x_5$ be the 5 observations.

Then

$$\frac{x_1 + x_2 + \cdots + x_5}{5} = 12$$ (05)

and

$$\frac{x_1^2 + x_2^2 + \cdots + x_5^2}{5} - 12^2 = 2\,.$$ (05)

$$\therefore x_1^2 + x_2^2 + \cdots + x_5^2 = 730$$

Let x_6 be the 6^{th} observation.

Then,

$$\frac{x_1 + \cdots + x_5 + x_6}{6} = 11\,.$$

$$\therefore x_6 = 66 - (x_1 + \cdots + x_5)$$

$$= 66 - 60 = 6$$ (05)

New variance $= \frac{730+36}{6} - 11^2$ (05)

$= \frac{766}{6} - 121 = \frac{383-363}{3} = \frac{20}{3}$ (05)

25

5. Heights of a group of persons are normally distributed with a mean of 58 inches and a standard deviation of 4 inches. Find the probability that the height of a randomly selected person from this group is
(i) at most 60 inches,
(ii) at least 56 inches, given that the person is shorter than 60 inches.

Let X denote the height of a randomly selected person from this group.

$$X \sim N(58, 4^2)$$

(i)

$$P(X \leq 60) = P\left(\frac{X-58}{4} \leq \frac{60-58}{4}\right) \quad \text{(05)}$$

$$= P(z < 0.5)$$

$$= 0.5 + P(0 < z < 0.5)$$

$$= 0.5 + 0.1915$$

$$= 0.6915 \quad \text{(05)}$$

(ii)

$$P(X > 56 \mid X < 60) = \frac{P(56 < X < 60)}{P(X < 60)} \quad \text{(05)}$$

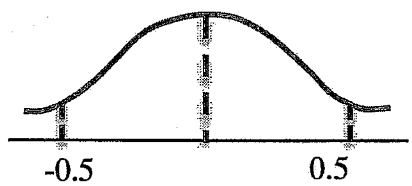
$$P(56 < X < 60) = P\left(\frac{56-58}{4} < X < \frac{60-58}{4}\right) \quad \text{(05)}$$

$$= P(-0.5 < z < 0.5)$$

$$= 2P(0 < z < 0.5)$$

$$= 2 \times 0.1915$$

$$= 0.3830 \quad \text{(05)}$$



$$\therefore \text{The answer} = \frac{0.3830}{0.6915} = 0.5534$$

25

6. The cards in a pack of five cards are labelled as 1, 2, 3, 4 and 5. Cards are randomly drawn from the pack, with replacement, until an even numbered card is obtained. Find the probability that an even numbered card is obtained
(i) at the first draw,
(ii) before the third draw,
(iii) at the third draw.

The five cards have labels $1, 2, 3, 4, 5$.

Let X denote the number of draws required until an even-numbered card is obtained.

Then $X \sim \text{geom}(p)$, where $p = \frac{2}{5} = 0.4$. (05)

(05) or

$$P(X = x) = (1 - p)^{x-1}p, \quad x = 1, 2, 3, \dots$$

(i) $P(X = 1) = p = \frac{2}{5} = 0.4$. (05)

(ii) $P(X < 3) = P(X = 1) + P(X = 2)$,

Since there are mutually exclusive events.

$$P(X = 1) = 0.4$$
$$P(X = 2) = (1 - p)p = 0.6 \times 0.4 = 0.24.$$

Thus, $P(X < 3) = 0.4 + 0.24 = 0.64$. (05)

(iii) $P(X = 3) = (1 - p)^2 p$
$= (0.6)^2 \times 0.4 = 0.36 \times 0.4$
$= 0.144$ (05)

25

7. The probability that a participant successfully completes Task 1 and Task 2 assigned in a competition are 0.80 and 0.68 respectively. The probability that a participant who is successful in Task 1 will also be successful in Task 2 is 0.7. Find the probability that a participant who is not successful in Task 1 will be successful in Task 2.
Also, find the probability that among three randomly selected participants, no one successfully completes Task 1.

$P(\text{Successful in Task 1}) = 0.80$

$P(\text{Successful in Task 2}) = 0.68$

$P(\text{Successful in Task 2} \mid \text{Successful in Task 1}) = 0.7$

Let $p = P(\text{Successful in Task 2} \mid \text{Not successful in Task 1})$.

From the total probability law, we write

$$P(\text{Successful in Task 2})$$
$$= P(\text{Successful in Task 2} \mid \text{Successful in Task 1}) \times P(\text{Successful in Task 1}) + P(\text{Successful in Task 2} \mid \text{Not successful in Task 1}) \times P(\text{Not successful in Task 1}).$$ (05)

Thus,

$$0.68 = 0.7 \times 0.8 + p \times (1 - 0.8)$$
$$= 0.56 + 0.2p$$ (05)

$$\Rightarrow 0.2p = 0.12$$
$$p = \frac{0.12}{0.20} = \frac{3}{5}$$
$$= 0.6$$ (05)

Let X denote the number of participants who successfully complete Task 1, among the three randomly selected participants.

Then $X \sim \text{bin}(3, p')$, where $p' = P(\text{Successful in Task 1})$
$= 0.8$ (05)

$$P(X = 0) = {}^3C_0 (0.8)^0 (1 - 0.8)^3$$
$$= (0.2)^3 = 0.008$$ (05)

25

8. Let A and B be two events on a sample space S such that $P(A)=0.4$, $P\left((A\cup B)'\right)=0.2$ and $P(A\cap B)=0.1$. Find $P(B)$, $P(A\cap B')$ and $P(A\mid B)$.

Let A and B be two events defined on a sample space S such that $P(A) = 0.4$, $P((A \cup B)') = 0.2$ and $P(A \cap B) = 0.1$.

Note that,

$$P(A \cup B) = 1 - P((A \cup B)') \quad (05)$$

$$= 1 - 0.2 = 0.8.$$

Now

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad (05)$$

$$\Rightarrow 0.8 = 0.4 + P(B) - 0.1$$

$$\Rightarrow P(B) = 0.5 \quad (05)$$

We can write the event A as,

$$A = (A \cap B') \cup (A \cap B)$$

$$P(A) = P(A \cap B') + P(A \cap B)$$

(Since the two events are mutually exclusive.)

Thus,

$$P(A \cap B') = P(A) - P(A \cap B)$$
$$= 0.4 - 0.1$$

$$= 0.3 \quad (05)$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$
$$= \frac{0.1}{0.5}$$

$$= 0.2 \quad (05)$$

25

9. The probability mass function of a discrete random variable X is given below:

x	0	1	2	3
$P(X=x)$	a	b	0.3	0.1

The expected value of X is 1.3. Find the values of a and b.
Also, find $P(X \geq 2 \mid X \geq 1)$.

Let X be a discrete random variable with the following probability mass function.

x	0	1	2	3
$P(X=x)$	a	b	0.3	0.1

From the probability mass function

$$\sum_{x=0}^{3} P(X=x) = 1$$

$$\Rightarrow a + b + 0.4 = 1 \quad (05)$$

$$a + b = 0.6$$

Expected value of $X = E(X) = 1.3$

$$\Rightarrow \sum_{x=0}^{3} x\,P(X=x) = 1.3$$

$$\Rightarrow 0 + b + 0.6 + 0.3 = 1.3 \quad (05)$$

$$\Rightarrow b = 0.4$$

Thus, $a = 0.2,\ b = 0.4$ (05)

$$P(X \geq 2 \mid X \geq 1) = \frac{P\big((X \geq 2) \cap (X \geq 1)\big)}{P(X \geq 1)} \quad (05)$$

$$= \frac{P(X \geq 2)}{P(X \geq 1)} = \frac{0.4}{0.8}$$

$$= 0.5$$

(05)

25

10. A continuous random variable X has the probability density function given by

$$f(x)=\begin{cases} 2-kx & , \text{ for } 0\leq x\leq 1 \\ 0 & , \text{ otherwise} \end{cases}$$

Find the values of k and $E(X)$.

Also, find $P(X\leq\frac{1}{4})$.

The probability density function of the continuous random variable X is given by

$$f(x)=\begin{cases} 2-kx, & 0\leq x\leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Since $f(x)$ is a probability density function, we find

$$\int_0^1 f(x)\,dx = 1$$

$$\Rightarrow \int_0^1 (2-kx)\,dx = 1$$

$$\Rightarrow 2[x]_0^1 - \frac{k}{2}[x^2]_0^1 = 1$$

$$\Rightarrow 2-\frac{k}{2}=1$$

$$\Rightarrow k=2 \quad (05)$$

$$E(X)=\int_0^1 x\,f(x)\,dx \quad (05)$$

$$=\int_0^1 x\,(2-kx)dx = 2\int_0^1 x\,(1-x)dx$$

$$=[x^2]_0^1-\frac{2}{3}[x^3]_0^1$$

$$=1-\frac{2}{3}=\frac{1}{3} \quad (05)$$

$$P\left(X\leq\frac{1}{4}\right)=\int_0^{\frac{1}{4}}(2-2x)\,dx \quad (05)$$

$$=2[x]_0^{\frac{1}{4}}-[x^2]_0^{\frac{1}{4}}$$

$$=2\times\frac{1}{4}-\frac{1}{16}$$

$$=\frac{7}{16} \quad (05)$$

$$=0.4375$$

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Part B

11. A manufacturer produces a certain product in two qualities: Grade I and Grade II. The same raw materials are used to produce both types. The amount of raw materials and labour hours needed to produce one unit in each quality and the profit per unit in each quality are shown in the table below:

	Per unit values	
	Grade I product	Grade II product
Raw materials (kg)	8	10
Labour hours	6	5
Profit (Rs)	140	100

According to the market demand, at least 60 units from each quality must be produced per day.

On each day, the manufacturer has 2 400 kg of raw material and 1 500 labour hours.

It is required to find the number of units to be produced per day from each quality to maximize the total profit.

(i) Formulate this as a linear programming problem.

(ii) Sketch the feasible region.

(iii) Using the graphical method, find the optimum solution for this problem.

(iv) During the production of 2 units of Grade II, if 1 unit of a by-product is obtained that gives a profit of Rs. 80 per unit, find the optimum solution under this new situation.

Let x and y be the number of units to be produced per day from grade I quality products and grade II quality products, respectively.

(i) Maximize :

$$z = 140x + 100y \quad (10)$$

subject to :

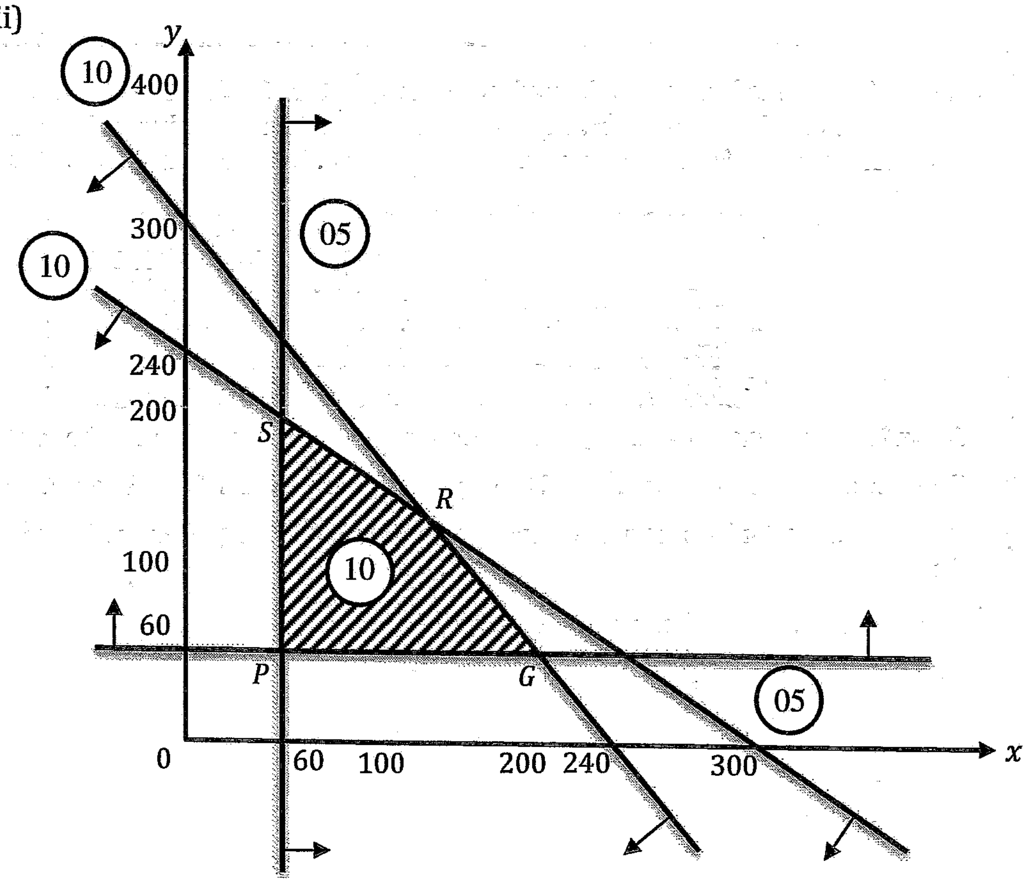
$$8x + 10y \leq 2400 \quad (10)$$

$$6x + 5y \leq 1500 \quad (10)$$

$$x \geq 60, \quad y \geq 60 \quad (05) + (05)$$

40

(ii)



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(iii)

Point	$z = 140x + 100y$
$P \equiv (60, 60)$	14400
$Q \equiv (200, 60)$	34000
$R \equiv (150, 120)$	33000
$S \equiv (60, 192)$	27600

(05) + (05)
(05) + (05)
(05) + (05)
(05) + (05)

The optimum solution: $x = 200$ and $y = 60$. (05)

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(iv) The new objective function:

$$z_1 = 140x + 100y + 80 \times \frac{y}{2} \quad (05)$$
$$= 140x + 140y$$

Point	$z = 140x + 140y$
$P \equiv (60,60)$	16800
$Q \equiv (200,60)$	36400
$R \equiv (150,120)$	37800
$S \equiv (60,192)$	35280

(15)

The new optimum solution: $x = 150$ and $y = 120$. (05)

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12.(a) Let $\mathbf{A} = \begin{pmatrix} 3 & a \\ -4 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & b \\ 0 & 2 \\ 1 & 1 \end{pmatrix}$, where $a, b \in \mathbb{R}$.

Find the values of a for which $\mathbf{A}^{-1}$ exists.

Now, let $a = -2$. Write down $\mathbf{A}^{-1}$.

Let $\mathbf{C} = \mathbf{BA}^{-1}$. Find $\mathbf{C}$ in terms of b.

Find the value of b such that $\mathbf{C}^{\mathrm{T}}\mathbf{B} = \begin{pmatrix} 39 & 15 \\ 25 & 12 \end{pmatrix}$.

(b) Let $a, b \in \mathbb{R}$. Write down the system of linear equations

$$ax - y = 2$$
$$4x - 2y = b$$

in the form $\mathbf{AX} = \mathbf{B}$, where $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$, and $\mathbf{A}$ and $\mathbf{B}$ are matrices to be determined.

Show that the above system of equations has

(i) a unique solution, when $a \neq 2$,

(ii) infinitely many solutions, when $a = 2$ and $b = 4$,

(iii) no solutions, when $a = 2$ and $b \neq 4$.

(a) Let $A = \begin{pmatrix} 3 & a \\ -4 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & b \\ 0 & 2 \\ 1 & 1 \end{pmatrix}$, $a, b \in \mathbb{R}$.

For A^{-1} to exists, $\det(A) \neq 0$. (05)

This gives $9 + 4a \neq 0$. (05)

Thus, $a < -\frac{9}{4}$ or $a > -\frac{9}{4}$ (05)

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Let $a = -2$. Then $A = \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix}$ and $\det(A) = 1$. (05)

$$A^{-1} = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$$ (10)

15

Let $C = BA^{-1} = \begin{pmatrix} 4 & b \\ 0 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$

$= \begin{pmatrix} 12+4b & 8+3b \\ 8 & 6 \\ 7 & 5 \end{pmatrix}$ (30)

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$C^T B = \begin{pmatrix} 12+4b & 8 & 7 \\ 8+3b & 6 & 5 \end{pmatrix} \begin{pmatrix} 4 & b \\ 0 & 2 \\ 1 & 1 \end{pmatrix}$ (10)

$= \begin{pmatrix} 55+16b & 23+12b+4b^2 \\ 37+12b & 17+8b+3b^2 \end{pmatrix}$ (20)

Since $C^T B = \begin{pmatrix} 39 & 15 \\ 25 & 12 \end{pmatrix}$, we find

$$55 + 16b = 39$$

$$16b = -16$$

$$b = -1 \quad (10)$$

This value of b satisfies the other 3 entries also.

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(b) Let $a, b \in \mathbb{R}$.

$$ax - y = 2$$

$$4x - 2y = b$$

Let $A = \begin{pmatrix} a & -1 \\ 4 & -2 \end{pmatrix}$, (10) $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $B = \begin{pmatrix} 2 \\ b \end{pmatrix}$ (05)

Then $AX = B$. (05)

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(i) $\det(A) = -2a + 4$ (05)

$a \neq 2$ $\det(A) \neq 0$.

$\therefore$ The system of equation has a unique solution. (05)

(ii) $a = 2$ and $b = 4$

The system becomes $2x - y = 2$. (05)

Let $x = t$, where $t \in \mathbb{R}$.

Then $y = 2t - 2$.

$\therefore$ The system has infinitely many solutions. (05)

(iii) $a = 2$ and $b \neq 4$

The system: $2x - y = 2$

$$2x - y = \frac{b}{2} \neq 2$$ (05)

$\therefore$ The system is inconsistent, and it has no solution. (05)

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13. (*a*) An unbiased die with faces marked 1, 1, 2, 3, 4, 4 is rolled and an unbiased spinner with six separated regions numbered 1, 2, 3, 2, 2, 1 is spun.

The number on the top face of the die and the number on the region pointed by the arrow of the spinner are recorded.

Let A be the event that both recorded numbers are the same and B be the event that the sum of the two recorded numbers is even.

Find $P(A')$, $P(A' \cap B)$ and $P(A' \cup B)$.

(*b*) Numbers consisting of 5 digits are made by selecting digits from the seven digits from 1 to 7 without repeating any digit.

Find

(i) the total number of different numbers that can be made,

(ii) the number of different numbers that can be made starting with 3, and having 6 and 7 adjacent to each other.

(*c*) A mathematics society consists of 9 senior members and 6 junior members. A research team of 5 people is to be formed to work on a new project.

Find the number of different ways in which the research team can be made

(i) out of all members,

(ii) such that the team has at least 3 senior members.

(a) Die: 1, 1, 2, 3, 4, 4

Spinner: 1, 2, 3, 2, 2, 1

The outcomes with the number on the die x and the number on the spinner y be denoted by $(x,\ y)$.

Then, A consists of the elements $(1,1), (2,2)$ and $(3,3)$.

$$P(1,1) = \frac{2}{6} \times \frac{2}{6} = \frac{1}{9} \quad (05)$$

$$P(2,2) = \frac{1}{6} \times \frac{3}{6} = \frac{1}{12} \quad (05)$$

$$P(3,3) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36} \quad (05)$$

$$\therefore P(A) = \frac{1}{9} + \frac{1}{12} + \frac{1}{36} = \frac{8}{36} = \frac{2}{9} \quad (10)$$

$$\therefore P(A') = 1 - P(A) = \frac{7}{9} \quad (05)$$

$A' \cap B$ consists of the element $(1,3), (3,1)$ and $(4,2)$.

$$\therefore P(A' \cap B) = P(1,3) + P(3,1) + P(4,2) \quad (10)$$

$$= \frac{2}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{2}{6} + \frac{2}{6} \times \frac{3}{6} \quad (10)$$

$$= \frac{10}{36} = \frac{5}{18} \quad (05)$$

Note that $A \cap B' = \emptyset$ (05)

$\therefore\ A' \cup B = (A \cap B')' = \emptyset' = S$, where S is the sample space.

$$\therefore P(A' \cup B) = P(S) = 1. \quad (05)$$

65

(b)

(i) The total number of different numbers that can be made $= {}^{7}P_{5}$ (10)

$$= \frac{7!}{2!}$$

$$= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1}$$

$$= 2520 \quad (05)$$

15

(ii) Since the number starts with 3, the first digit is fixed. That leaves 4 positions to be filled. Since 6 and 7 must be adjacent, we treat them a single block. Within the block, 6 and 7 can be arranged in 2 ways.

After choosing 3, 6 and 7, the remaining available digits are $1, 2, 4$ and 5.

The number of ways to choose 2 digits from 4 $= {}^{4}C_{2} = 6.$ (10)

The number of ways to arrange these 2 digits and the $\boxed{6,7}$ block $= 3!$ (10)

The answer $= 2 \times 6 \times 3! = 72.$ (10)

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(c)

(i) The answer $= {}^{15}C_5$ (10)

$$= \frac{15!}{5! \times 10!}$$

$$= \frac{15 \times 14 \times 13 \times 12 \times 11}{5 \times 4 \times 3 \times 2 \times 1}$$

$$= 7 \times 13 \times 3 \times 11$$

$$= 3003$$ (10)

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(ii) The number of different ways with:

3 seniors $= {}^9C_3 \times {}^6C_2$

$$= \frac{9 \times 8 \times 7}{3 \times 2 \times 1}$$

$$= 1260$$ (05)

4 seniors $= {}^9C_4 \times {}^6C_1$

$$= \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1}$$

$$= 756$$ (05)

5 seniors $= {}^9C_5$

$$= \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1}$$

$$= 126$$ (05)

The answer $= 1260 + 756 + 126 = 2142.$ (05)

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14. The probability that a school bus arrives on time on a Monday is 0.7. If the bus arrives on time on a particular day, the probability that it will be on time on the following day is 0.8. If the bus is late on a particular day, the probability that it will be late on the following day is 0.4.

Find the probability that the bus arrives on time on a Wednesday.

Also, find the probability that, from Monday to Wednesday of a week, the bus will be

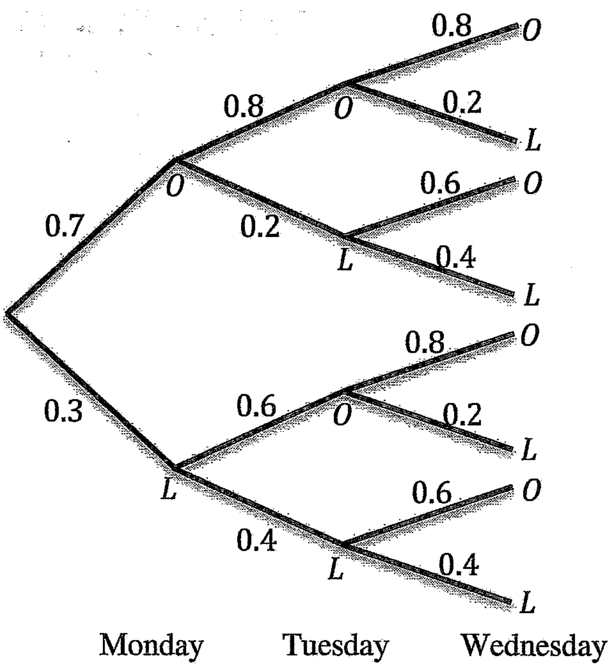
(*a*) late at least on one day,

(*b*) late exactly on two days,

(*c*) late exactly on two days, given that it was late at least on one day.

Let L^D denote the event that the bus arrives late on a particular day, D.

Let O^D denote the event that the bus arrives on time on a particular day, D.



$P(O^W) = P(O^W|O^T)P(O^T) + P(O^W|L^T)P(L^T)$ (10)

$P(O^T) = P(O^T|O^M)P(O^M) + P(O^T|L^M)P(L^M)$ (10)

$= 0.8 \times 0.7 + 0.6 \times 0.3$ (10)

$= 0.56 + 0.18$

$= 0.74$ (05)

$P(L^T) = 1 - P(O^T) = 1 - 0.74 = 0.26$ (10)

$\therefore P(O^W) = 0.8 \times 0.74 + 0.6 \times 0.26$ (10)

$= 0.592 + 0.156$

$= 0.748$ (05)

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(a) $P(\text{Bus is late at least on one day}) = 1 - P(\text{Bus is on time on all 3 days})$ (10)

$= 1 - P(O^W \cap O^T \cap O^M)$

Now,

$P(O^W \cap O^T \cap O^M) = P(O^W \mid O^T \cap O^M) \cdot P(O^T \cap O^M)$ (10)

$= P(O^W \mid O^T \cap O^M) \cdot P(O^T|O^M) \cdot P(O^M)$ (05)

$= 0.8 \times 0.8 \times 0.7$ (10)

$= 0.448$ (05)

$\therefore P(\text{Bus is late at least on one day}) = 1 - 0.448$

$= 0.552$ (05)

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(b) $P(\text{Bus will be late exactly on two days})$

$= P(L^M \cap L^T \cap O^W) + P(L^M \cap O^T \cap L^W) + P(O^M \cap L^T \cap L^W)$ (10)

$= 0.3 \times 0.4 \times 0.6 + 0.3 \times 0.6 \times 0.2 + 0.7 \times 0.2 \times 0.4$ (10)

$= 0.072 + 0.036 + 0.056$

$= 0.164$ (05)

25

(c) Let A be the event that the bus is late at least on one day.
From part (a), $P(A) = 0.552$.

Let B be the event that the bus is late exactly on two days.
From part (b), $P(B) = 0.164$.

Need

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} \quad \text{(05)}$$

$$= \frac{P(B)}{P(A)} \quad \text{(05)}$$

Since $B \subseteq A, P(A \cap B) = P(B)$

$$= \frac{0.164}{0.552} \quad \text{(05)}$$

$$\approx 0.297 \quad \text{(05)}$$

20

(ii) $P(B) = P(48 \leq X \leq 52)$

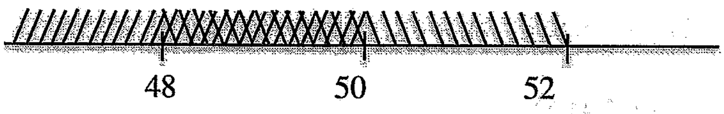
$$= P\left(\frac{48-48}{4} \leq \frac{X-48}{4} \leq \frac{52-48}{4}\right) \quad (05)$$

$$= P(0 \leq Z \leq 1) \quad (05)$$

$$= 0.3413 \quad (05)$$

15

(iii)



$$A \cap B' = \{X < 48\} \quad (10)$$

$$P(A \cap B') = P\left(\frac{X-48}{4} < \frac{48-48}{4}\right) \quad (05)$$

$$= P(Z < 0) \quad (05)$$

$$= 0.5 \quad (05)$$

25

(iv)

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad (05)$$

$$A \cap B = \{48 \leq X < 50\} \quad (10)$$

$$P(A \cap B) = P\left(\frac{48-48}{4} \leq \frac{X-48}{4} < \frac{50-48}{4}\right) \quad (05)$$

$$= P(0 \leq Z < 0.5) \quad (05)$$

$$= 0.1915 \quad (05)$$

From part (ii),

$$P(B) = 0.3413$$

Thus,

$$P(A|B) = \frac{0.1915}{0.3413} = 0.5611 \quad (05)$$

(10)

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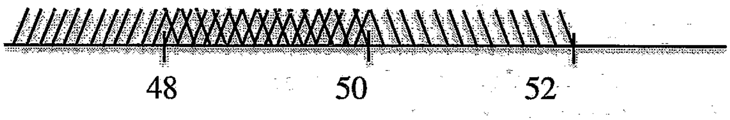
(v)



$$A \cup B = \{X \leq 52\} \quad (10)$$

$$P(A \cup B) = P(X \leq 52)$$

$$= P\left(\frac{X-48}{4} \leq \frac{52-48}{4}\right) \quad (05)$$

$$= P(Z \leq 1) \quad (05)$$

$$= 0.5 + P(0 < Z < 1)$$

$$= 0.5 + 0.3413$$

$$= 0.8413 \quad (10)$$

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16. The following table summarises the distances travelled by a taxi driver during a certain period of time:

Distance (km)	No. of days
10 – 20	9
20 – 30	13
30 – 40	a
40 – 50	16
50 – 60	15
60 – 70	5

(i) If the cumulative frequency corresponding to the 4th class interval is 60, find the value of a.

(ii) Using the transformation $y_i = \frac{x_i - 35}{10}$, or otherwise estimate

(*a*) total distance travelled, during this period of time,

(*b*) mean of the distance travelled per day,

(*c*) median of the distances travelled during this period of time.

(iii) Based on the values of the mean and the median, what can be concluded about the shape of the distribution of the data?

(iv) Sketch the cumulative frequency curve.

(v) Using the curve sketched in part (iv), or otherwise, estimate

(*a*) the tenth percentile,

(*b*) the first quartile,

(*c*) the inter-quartile range

for the given data.

(i) Cumulative frequency corresponding to the 4^{th} class interval = 60

Thus, $9 + 13 + a + 16 = 60$ (10)

$\Rightarrow \quad a = 22.$ (05)

15

(ii)

Let $y_i = \frac{x_i - 35}{10}$

Distance (km)	Frequency (f_i)	Class midpoint	Midpoint for y values (m_i)	$f_i m_i$
10 – 20	9	15	−2	−18
20 – 30	13	25	−1	−13
30 – 40	22	35	0	0
40 – 50	16	45	1	16
50 – 60	15	55	2	30
60 – 70	5	65	3	15
	$\sum f_i = 80$	(10)	(10)	(10)

a)

$$\text{Total of } y_i \text{ values} = \sum_{i=1}^{80} y_i = \sum_{i=1}^{80} f_i m_i$$

$$= 30. \quad (10)$$

$$x_i = 35 + 10y_i$$

$$\sum_{i=1}^{80} x_i = 35 \times 80 + 10 \sum_{i=1}^{80} y_i \quad (10)$$

$$= 2800 + 300$$

$$\text{Total distance travelled} = 3100\ km \quad (05)$$

55

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b)

$$\text{Mean distance travelled per day} = \frac{3100}{80}\ km \text{ per day} \quad (05)$$

$$= 38.75\ km \text{ per day} \quad (05)$$

10

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c) Total number of days $= n = \sum f_i = 80$

Median $= 40^{th}$ data value

Median belongs to the third-class interval.

$$\text{Thus, median} = 30 + \frac{10}{22} \times (40 - 22) \quad (10)$$

$$= 30 + \frac{10}{22} \times 18$$

$$= 30 + \frac{90}{11}$$

$$= 30 + 8.18$$

$$= 38.18\ km \quad (05)$$

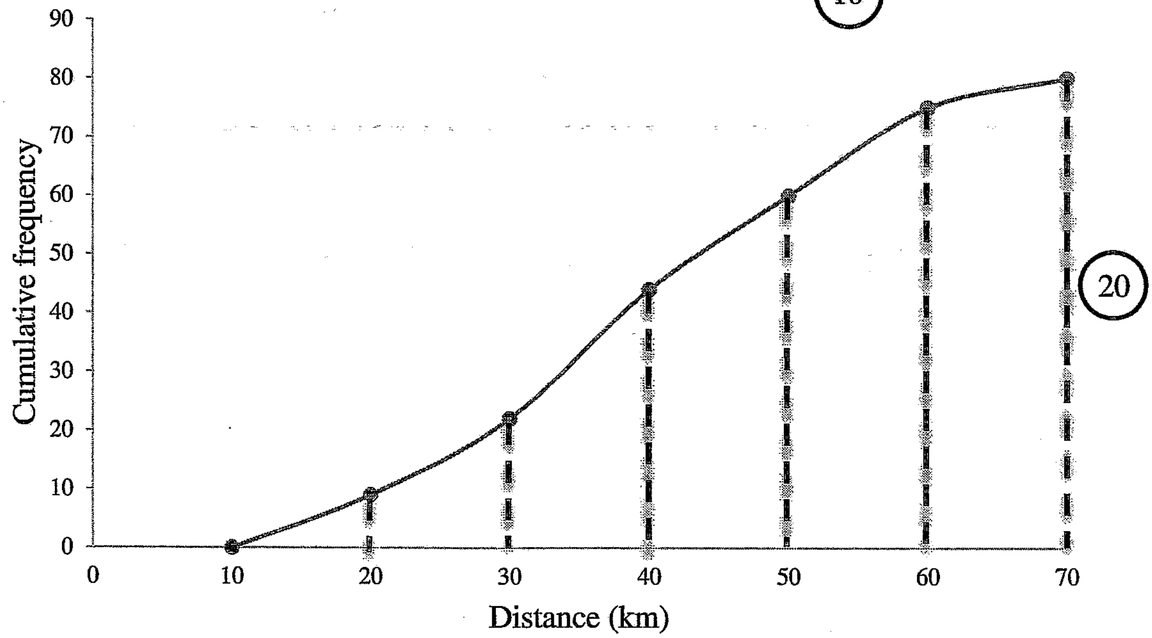
15

(iii) We find that the mean (= 38.75) and median (= 38.18) are close. Hence, the shape of the distribution is symmetric. (05)

05

Distance (km)	Frequency	Cumulative frequency
10 – 20	9	9
20 – 30	13	22
30 – 40	22	44
40 – 50	16	60
50 – 60	15	75
60 – 70	5	80

(10)



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(iv)

(a) Tenth percentile corresponds to $\frac{10}{100} \times 80 \Rightarrow 8^{th}$ observation.
From the graph, 10^{th} percentile $= 18\ km$. (05)

05

(b) First quartile $= 25^{th}$ percentile

$= \frac{25}{100} \times 80^{th}$ observation

$= 20^{th}$ observation

From the graph, First quartile $= 28\ km$. (05)

05

(c) Third quartile $= Q_3 = \frac{75}{100} \times 80^{th}$ observation

$= 60^{th}$ observation

From the graph, $Q_3 = 50$ (or from the table) (05)

Inter quartile range $= Q_3 - Q_1$

$= (50 - 28)\ km$

$= 22\ km$ (05)

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17. The duration of activities of a project and the flow of activities are given in the following table:

Activity	Preceding Activity/ Activities	Duration (in months)
A	–	3
B	–	4
C	A, B	5
D	A, C	2
E	C	3
F	B, C	4
G	D, E	5
H	E, F	6
I	G, H	1

(i) Construct the project network.

(ii) Prepare an activity schedule that indicates earliest start time, earliest finish time, latest start time, latest finish time and the float for each activity.

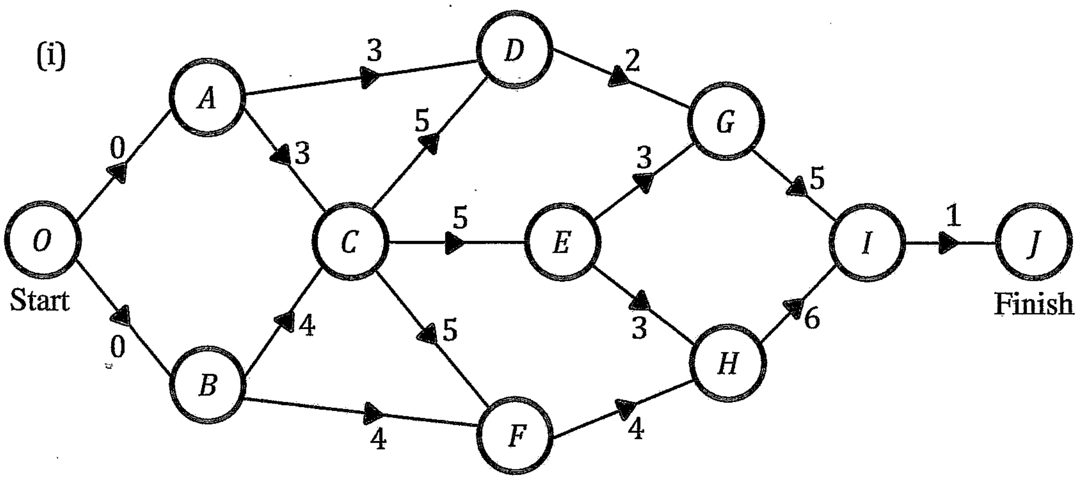
(iii) Find the total duration of the project.

(iv) Write down the critical path of the project.

(v) What are the activities that can be delayed without extending the total duration of the project?

(vi) How is the project completion time affected by delaying the activity F by 2 months?

(i)



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(ii)

Activity	ES	EF	LS	LF	Float/Slack
A	0	$0+3=3$	$4-3=1$	$1+3=4$	1
B	0	$0+4=4$	$4-4=0$	$0+4=4$	0
C	4	$4+5=9$	$9-5=4$	$4+5=9$	0
D	9	$9+2=11$	$14-2=12$	$12+2=14$	3
E	9	$9+3=12$	$13-3=10$	$10+3=13$	1
F	9	$9+4=13$	$13-4=9$	$9+4=13$	0
G	12	$12+5=17$	$19-5=14$	$14+5=19$	2
H	13	$13+6=19$	$19-6=13$	$13+6=19$	0
I	19	$19+1=20$	$20-1=19$	$19+1=20$	0

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(iii) The total duration of the project is $= 20$ days.

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(iv) The critical path is: $B \rightarrow C \rightarrow F \rightarrow H \rightarrow I$.

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(v) Activities that can be delayed without extending the duration of the project are,

A, D, E, G.

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(vi) The activity F is on the critical path.

$\therefore$ Delaying F by 2 months will delay the entire project by 2 months.

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