Department of Examinations - Sri Lanka

G.C.E. (A/L) Examination - 2024

10 – Combined Mathematics I

Marking Scheme

This has been prepared for the use of marking examiners. Changes would be made according to the views presented at the Chief/Assistant Examiners' meeting.

Amendments to be included.

G. C. E (Advanced Level) Examination – 2024

10 - Combined Mathematics I

Distribution of Marks

Paper I

Part A	=	10 × 25	=	250
Part B	=	05 × 150	=	750

Total $= \dfrac{1000}{10}$

Final marks = **100**

Common Techniques of Marking Answer Scripts.

It is compulsory to adhere to the following standard method in marking answer scripts and entering marks into the mark sheets.

1. Use a red color ball point pen for marking. (Only Chief/Additional Chief Examiner may use a mauve color pen.)
2. Note down Examiner's Code Number and initials on the front page of each answer script.
3. Write off any numerals written wrong with a clear single line and authenticate the alterations with Examiner's initials.
4. Write down marks of each subsection in a △ and write the final marks of each question as a rational number in a □ with the question number. Use the column assigned for Examiners to write down marks.

Example: **Question No. 03**

(i) ....
.... √ △ $\frac{4}{5}$
....

(ii) ....
.... √ △ $\frac{3}{5}$
....

(iii) ....
.... √ △ $\frac{3}{5}$
....

03 (i) $\frac{4}{5}$ + (ii) $\frac{3}{5}$ + (iii) $\frac{3}{5}$ = $\boxed{\frac{10}{15}}$

MCQ answer scripts: (Template)

1. Marking templets for G.C.E.(A/L) and GIT examination will be provided by the Department of Examinations itself. Marking examiners bear the responsibility of using correctly prepared and certified templates.
2. Then, check the answer scripts carefully. If there are more than one or no answers Marked to a certain question write off the options with a line. Sometimes candidates may have erased an option marked previously and selected another option. In such occasions, if the erasure is not clear write off those options too.
3. Place the template on the answer script correctly. Mark the right answers with a '√' and the wrong answers with a 'X' against the options column. Write down the number of correct answers inside the cage given under each column. Then, add those numbers and write the number of correct answers in the relevant cage.

Structured essay type and assay type answer scripts:

1. Cross off any pages left blank by candidates. Underline wrong or unsuitable answers. Show areas where marks can be offered with check marks.

2. Use the right margin of the overland paper to write down the marks.
3. Write down the marks given for each question against the question number in the relevant cage on the front page in two digits. Selection of questions should be in accordance with the instructions given in the question paper. Mark all answers and transfer the marks to the front page, and write off answers with lower marks if extra questions have been answered against instructions.

4. Add the total carefully and write in the relevant cage on the front page. Turn pages of answer script and add all the marks given for all answers again. Check whether that total tallies with the total marks written on the front page.

Preparation of Mark Sheets.

Except for the subjects with a single question paper, final marks of two papers will not be calculated within the evaluation board this time. Therefore, add separate mark sheets for each of the question paper. Write paper 01 marks in the paper 01 column of the mark sheet and write them in words too. Write paper II Marks in the paper II Column and wright the relevant details.

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1. Using the **Principle of Mathematical Induction**, prove that $7^n - 1$ is divisible by 6 for all $n \in \mathbb{Z}^+$.

For $n = 1$, $7^n - 1 = 7 - 1 = 6$, and it is divisible by 6.

$\therefore$ The result is true for $n = 1$. ⑤

Take any $k \in \mathbb{Z}^+$ and assume that the result is true for $n = k$.

i.e. $7^k - 1$ is divisible by 6. ⑤

$\therefore$ There exists $p \in \mathbb{Z}^+$ such that $7^k - 1 = 6p$. ⑤

Now, $7^{k+1} - 1 = 7.7^k - 1$

$= 7(6p + 1) - 1$

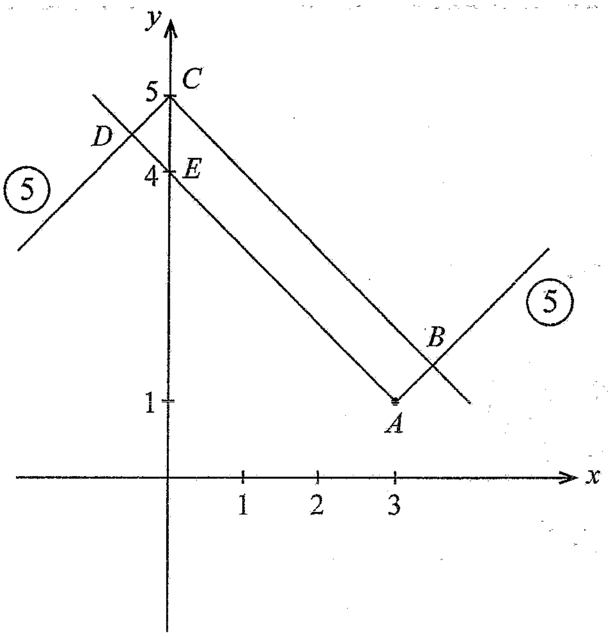
$= 42p + 6$

$= 6(7p + 1)$, and this is divisible by 6. ⑤

Hence, if the result is true for $n = k$, the it is also true for $n = k + 1$. The result has already been proved for $n = 1$.

Hence, by the principle of Mathematical Induction, the result is true for all $n \in \mathbb{Z}^+$. ⑤

2. Sketch the graphs of $y=|x-3|+1$ and $y=5-|x|$ in the same diagram.
Hence, find the area of the rectangular region enclosed by these graphs.



$CD = 1\cos 45° = \dfrac{1}{\sqrt{2}}$ ⑤

$AD = DE + EA = 1\sin 45° + \sqrt{3^2+3^2}$

$= \dfrac{1}{\sqrt{2}} + 3\sqrt{2}$ ⑤

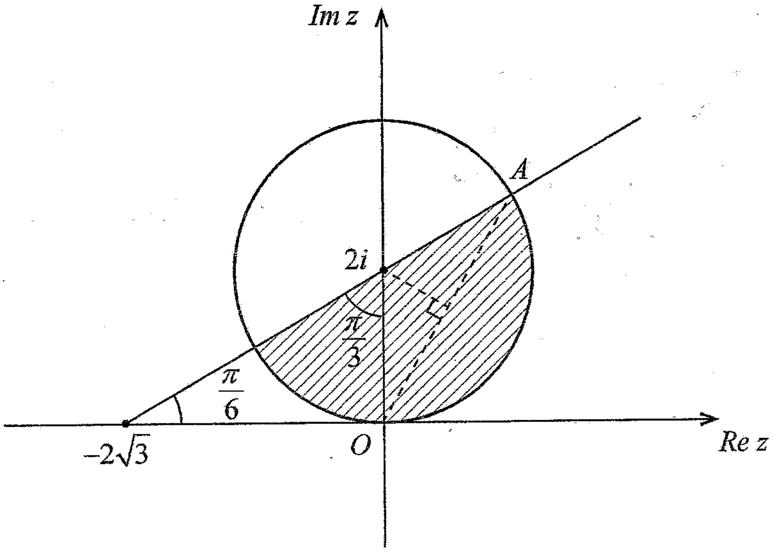
$= \dfrac{7}{\sqrt{2}}$

$\therefore$ The required area $= CD \cdot AD$

$= \dfrac{1}{\sqrt{2}} \cdot \dfrac{7}{\sqrt{2}}$

$= \dfrac{7}{2}$ area units. ⑤

3. Shade in an Argand diagram, the region consisting of points that represent the complex numbers z satisfying the inequalities $|z-2i|\leq 2$ and $0\leq \operatorname{Arg}(z+2\sqrt{3})\leq \frac{\pi}{6}$.
Find the greatest value of $|z|$ for the complex numbers z represented by the points in this shaded region.



Half line ⑤

Circle touching the x-axis at O ⑤

Shading the correct region ⑤

The required greatest value of $|z| = OA$ ⑤

$$= 2\times 2\sin\frac{\pi}{3}$$

$$= 2\sqrt{3}$$ ⑤

4. Show that the constant term in the expansion of $\left(1+x^3\right)\left(x-\frac{1}{\sqrt{x}}\right)^9$ is 93.

The general term of the binomial expansion of $\left(x-\frac{1}{\sqrt{x}}\right)^9 = {}^9C_r x^r\left(\frac{1}{\sqrt{x}}\right)^{9-r}$ ⑤

$$= {}^9C_r x^r \cdot x^{\left(\frac{r}{2}-\frac{9}{2}\right)}$$

$$= {}^9C_r\, x^{\left(\frac{3r}{2}-\frac{9}{2}\right)} \quad ⑤$$

$\therefore$ Constant terms occur in the expansion of $\left(1+x^3\right)\left(x-\frac{1}{\sqrt{x}}\right)^9$ when $\frac{3r}{2}-\frac{9}{2}=0$ and when $\frac{3r}{2}-\frac{9}{2}=3.$ ⑤

i.e. when $r=3$ and when $r=1$.

$\therefore$ The required constant term $= {}^9C_1 + {}^9C_3$ ⑤

$$= \frac{9!}{8!} + \frac{9!}{3!6!}$$

$$= 9 + \frac{9\times 8\times 7}{2\times 3}$$

$$= 9 + 84$$

$$= 93. \quad ⑤$$

5. Show that $\lim_{x\to 4}\dfrac{\left(\sqrt{x-3}-1\right)}{(x-4)^2}\sin\left(\sqrt{x}-2\right)=\dfrac{1}{8}$.

$$\lim_{x\to 4}\frac{\left(\sqrt{x-3}-1\right)}{(x-4)^2}\sin\left(\sqrt{x}-2\right)$$

$$=\lim_{x\to 4}\frac{\left(\sqrt{x-3}-1\right)}{(x-4)^2}\bullet\frac{\left(\sqrt{x-3}+1\right)}{\left(\sqrt{x-3}+1\right)}\bullet\sin\left(\sqrt{x}-2\right)$$ ⑤

$$=\lim_{x\to 4}\frac{(x-4)}{(x-4)^2\left(\sqrt{x-3}+1\right)}\bullet\sin\left(\sqrt{x}-2\right)$$

$$=\lim_{x\to 4}\frac{1}{\left(\sqrt{x}+2\right)\left(\sqrt{x-3}+1\right)}\bullet\frac{\sin\left(\sqrt{x}-2\right)}{\left(\sqrt{x}-2\right)}$$ ⑤

$$=\frac{1}{\left(\sqrt{4}+2\right)\left(\sqrt{1}+1\right)}\bullet 1$$ ⑤ + ⑤

$$=\frac{1}{8}.$$ ⑤

6. The region enclosed by the curves $y = \dfrac{2}{x\sqrt[4]{4-x^2}}$, $y = 0$, $x = 1$ and $x = \sqrt{2}$ is rotated about the x-axis through 2π radians. Show that the volume of the solid thus generated is $\pi\left(\sqrt{3}-1\right)$.

The required volume $= \pi \displaystyle\int_1^{\sqrt{2}} y^2 \, dx$ ⑤

$$= \pi \int_1^{\sqrt{2}} \frac{4}{x^2\sqrt{4-x^2}} dx$$

Let $x = 2\sin t$.
Then $dx = 2\cos t \, dt$.

$$= \pi \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{4}{4\sin^2 t\sqrt{4-4\sin^2 t}} \cdot 2\cos t \, dt$$ ⑤

$$= \pi \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\sin^2 t \cdot 2\cos t} \cdot 2\cos t \, dt$$

$$= \pi \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec}^2 t \, dt$$ ⑤

$$= \pi(-\cot t)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$ ⑤

$$= \pi\left(-\cot\frac{\pi}{4} + \cot\frac{\pi}{6}\right)$$

$$= \pi\left(\sqrt{3}-1\right).$$ ⑤

7. Let C be the curve given parametrically by $x = \ln t$ and $y = e^t + t\ln t$ for $t > 0$.
Show that $\frac{dy}{dx} = t(e^t + \ln t + 1)$.
If the tangent drawn to the curve C at the point corresponding to $t = 1$, passes through the point $(1, a)$, show that $a = 1 + 2e$.

$x = \ln t$ and $y = e^t + t\ln t$ for $t > 0$.

$\frac{dx}{dt} = \frac{1}{t}$ and $\frac{dy}{dt} = e^t + \ln t + t\frac{1}{t}$ ⑤

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{(e^t + \ln t + 1)}{\frac{1}{t}}$$

$$= t(e^t + \ln t + 1).$$ ⑤

$\left.\frac{dy}{dx}\right|_{t=1} = e + \ln 1 + 1 = e + 1.$ ⑤

The point corresponding to $t = 1$ is $(0, e)$. ⑤

$\therefore \frac{a-e}{1-0} = e + 1$

$\therefore a = 1 + 2e.$ ⑤

8. Find the equations of the two straight lines passing through the point $A \equiv (-1, 2)$ having a perpendicular distance of 1 from the origin.

The equation of any line through $A \equiv (-1, 2)$ is of the form $a(x + 1) + b(y - 2) = 0$, where $a, b \in \mathbb{R}$ with $a^2 + b^2 \neq 0$. ⑤

Given: $\dfrac{|a - 2b|}{\sqrt{a^2 + b^2}} = 1$ ⑤

$\Leftrightarrow \quad (a - 2b)^2 = a^2 + b^2$

$\Leftrightarrow \quad 4ab - 3b^2 = 0$

$\Leftrightarrow \quad b(4a - 3b) = 0$

$\Leftrightarrow \quad b = 0 \quad \text{or} \quad a = \dfrac{3b}{4}$ ⑤

The required equations are $a(x + 1) = 0$ or $\dfrac{3b}{4}(x + 1) + b(y - 2) = 0$.

So, $x = -1$ or $3x + 4y - 5 = 0$.
⑤ ⑤

Aliter

$x = -1$ is one such line through $A \equiv (-1, 2)$. ⑤

Let $y = mx + c$ be any other line through $A \equiv (-1, 2)$.

Then $2 = m(-1) + c$ and so $c = m + 2$.

$\therefore \ y = mx + m + 2.$ ⑤

Given: $\dfrac{|m + 2|}{\sqrt{m^2 + 1}} = 1$ ⑤

$\Leftrightarrow \quad (m + 2)^2 = m^2 + 1$

$\Leftrightarrow \quad 4m + 4 = 1$

$\Leftrightarrow \quad m = -\dfrac{3}{4}.$ ⑤

The other line is given by $y = \dfrac{3}{4}x + \dfrac{3}{4} \ 2.$

i.e. $3x + 4y - 5 = 0.$ ⑤

9\. Let $A \equiv (-1, 1)$ and $B \equiv (3, 3)$. Write down the equation of the circle S with AB as a diameter. Show that the circle $x^2+y^2-4x-5y+9=0$ touches the circle S internally at B.

The equation of S is

$$(x+1)(x-3)+(y-1)(y-3)=0. \quad ⑤$$

Let C_1 be the centre of S.

$$\left.\begin{array}{l}\text{Then } C_1 \equiv (1, 2). \\ \text{Its radius } r_1 = \sqrt{5}\end{array}\right\} ⑤$$

Let C_2 be the centre of $x^2+y^2-4x-5y+9=0$.

$$\left.\begin{array}{l}\text{Then } C_2 \equiv (2, \frac{5}{2}). \\ \text{Its radius } r_2 = \sqrt{(-2)^2+\left(\frac{-5}{2}\right)^2-9}=\frac{\sqrt{5}}{2}.\end{array}\right\} ⑤$$

$$\text{Now, } C_1C_2 = \sqrt{1^2+\left(\frac{1}{2}\right)^2}=\frac{\sqrt{5}}{2}.$$

$$\therefore\ C_1C_2 = r_1 - r_2. \quad ⑤$$

For the verification that B lies on the second circle. ⑤

$\therefore$ The circle $x^2+y^2-4x-5y+9=0$ touches the circle S internally at B.

10. Show that $\dfrac{\cot\theta}{1+\sin\theta}+\dfrac{\cot\theta}{1-\sin\theta}=4\operatorname{cosec}2\theta$.

Hence, solve $\dfrac{\cot\theta}{1+\sin\theta}+\dfrac{\cot\theta}{1-\sin\theta}=8\cos 2\theta$.

$$\frac{\cot\theta}{1+\sin\theta}+\frac{\cot\theta}{1-\sin\theta}$$

$$=\frac{[(1-\sin\theta)+(1+\sin\theta)]\cot\theta}{1-\sin^2\theta}$$

$$=\frac{2\cot\theta}{\cos^2\theta}$$ ⑤

$$=\frac{2}{\sin\theta\cos\theta}$$

$$=4\operatorname{cosec}2\theta.$$ ⑤

$$\frac{\cot\theta}{1+\sin\theta}+\frac{\cot\theta}{1-\sin\theta}=8\cos 2\theta$$

$$4\operatorname{cosec}2\theta=8\cos 2\theta$$ ⑤

$$2\cos 2\theta\sin 2\theta=1$$

$$\sin 4\theta=1$$ ⑤

$$\sin 4\theta=\sin\frac{\pi}{2}$$

$\therefore 4\theta=n\pi \quad (+1)^n\dfrac{\pi}{2}$, where $n\in\mathbb{Z}$.

$\therefore \theta=\dfrac{n\pi}{4} \quad (+1)^n\dfrac{\pi}{8}$, where $n\in\mathbb{Z}$. ⑤

11. (*a*) Let $f(x) = x^2 + 2x + c$, where $c \in \mathbb{R}$.

It is given that the equation $f(x) = 0$ has two real distinct roots. Show that $c < 1$.

Let α and β be the roots of $f(x) = 0$.

Show that $\alpha^2 + \beta^2 = 4 - 2c$.

Let $c \neq 0$ and $\lambda \in \mathbb{R}$. The quadratic equation with $\alpha + \frac{1}{\alpha}$ and $\beta + \frac{1}{\beta}$ as its roots is $2x^2 + 12x + \lambda = 0$. Find the values of c and λ.

(*b*) Let $f(x) = x^3 + px^2 + qx + p$, where $p, q \in \mathbb{R}$. The remainder when $f(x)$ is divided by $(x - 2)$ is 36 more than the remainder when $f(x)$ is divided by $(x - 1)$. Show that $3p + q = 29$.

It is also given that $(x + 1)$ is a factor of $f(x)$. Show that $p = 6$ and $q = 11$, and factorize $f(x)$ completely.

Hence, solve $f(x) = 3(x + 2)$.

⑤ ⑤

(*a*) $\Delta = 4 - 4c > 0$.

$\therefore\ c < 1$. ⑤ $\boxed{15}$

$\alpha + \beta = -2$ and $\alpha\beta = c$.

⑤ ⑤

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$ ⑤

$= 4 - 2c$. ⑤ $\boxed{20}$

$\left(\alpha + \frac{1}{\alpha}\right) + \left(\beta + \frac{1}{\beta}\right) = -\frac{12}{2}$ ⑩

$\alpha + \beta + \frac{\alpha + \beta}{\alpha\beta} = -6$ ⑤

$-2 - \frac{2}{c} = -6$ ⑤

$\frac{2}{c} = 4$

$c = \frac{1}{2}$ ⑤

$$\left(\alpha+\frac{1}{\alpha}\right)\left(\beta+\frac{1}{\beta}\right)=\frac{\lambda}{2}$$ ⑩

$$\alpha\beta+\frac{1}{\alpha\beta}+\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\lambda}{2}$$

$$\alpha\beta+\frac{1}{\alpha\beta}+\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{\lambda}{2}$$ ⑤

$$c+\frac{1}{c}+\frac{4-2c}{c}=\frac{\lambda}{2}$$

$$\frac{1}{2}+2+\frac{4-2\times\frac{1}{2}}{\frac{1}{2}}=\frac{\lambda}{2}$$ ⑤

$$\frac{5}{2}+6=\frac{\lambda}{2}$$

$\therefore\ \lambda = 17.$ ⑤

$\boxed{50}$

(b) $f(x) = x^3+px^2+qx+p$

The remainder when $f(x)$ is divided by $(x-2)$

$=f(2) = 8+4p+2q+p = 8+5p+2q.$ ⑤

The remainder when $f(x)$ is divided by $(x-1)$

$=f(1) = 1+p+q+p = 1+2p+q.$ ⑤

It is given that $f(2) = 36+f(1).$ ⑤

$8+5p+2q = 36+1+2p+q.$

$3p+q = 29$ ——— ①

⑤

$\boxed{20}$

Since $(x+1)$ is a factor of $f(x)$, we have $f(-1)=0.$ ⑤

$\therefore\ -1+p-q+p=0$

$\therefore\ 2p-q=1$ ——— ② ⑤

① and ② $\Rightarrow$ $5p=30$

$\therefore p = 6$ ⑤

Now ① $\Rightarrow$ $18+q=29$

$\therefore q = 11.$ ⑤

$f(x) = x^3 + 6x^2 + 11x + 6$

$= (x + 1)(x^2 + 5x + 6)$ ⑤

$= (x + 1)(x + 2)(x + 3)$ ⑤

$\boxed{30}$

$f(x) = 3(x + 2)$

$(x + 1)(x + 2)(x + 3) = 3(x + 2)$

$(x + 2)\,[(x + 1)(x + 3) - 3] = 0$ ⑤

$(x + 2)(x^2 + 4x) = 0$

$x(x + 2)(x + 4) = 0$ ⑤

$\therefore\ x = 0$ or $x = -2$ or $x = -4$. ⑤

$\boxed{15}$

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12. (*a*) The parents of a family decide to invite 6 out of 15 of their close relatives for a dinner. While the father has 5 close female relatives and 3 close male relatives, the mother has 3 close female relatives and 4 close male relatives.

Find the number of different ways in which

(i) the father can invite 3 of his close female relatives and the mother can invite 3 of her close male relatives,

(ii) the father can invite 3 of his close relatives and the mother can invite 3 of her close relatives so that 3 males and 3 females are invited.

(*b*) Let $U_r = \dfrac{1}{r(r+2)(r+4)}$ and $f(r) = \dfrac{1}{r(r+2)}$ for $r \in \mathbb{Z}^+$.

Determine the value of the real constant A such that $f(r) - f(r+2) = AU_r$ for $r \in \mathbb{Z}^+$.

Hence, show that $\sum\limits_{r=1}^{n} U_r = \dfrac{11}{96} - \dfrac{1}{4(n+1)(n+3)} - \dfrac{1}{4(n+2)(n+4)}$ for $n \in \mathbb{Z}^+$.

Show further that the infinite series $\sum\limits_{r=1}^{\infty} U_r$ is convergent and find its sum.

Find the value of the real constant m such that $\lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} (mU_r + U_{n+1-r}) = \dfrac{11}{32}$.

12. (*a*)

Father		Mother	
Female relatives	Male relatives	Female relatives	Male relatives
5	3	3	4

(i) The number of different ways, the father can invite 3 close female relatives = $^5C_3$ ⑤

The number of different ways, the mother can invite 3 close male relatives = $^4C_3$ ⑤

∴ The number of different ways, the father can invite 3 close female relatives and the mother can invite 3 close male relatives = $^5C_3 \times {}^4C_3$ ⑤

$$= \frac{5!}{3!2!} \times \frac{4!}{3!1!} = \frac{5 \times \overset{2}{\cancel{4}}}{\cancel{2}} \quad 4 \times 40 \qquad ⑤$$

$\boxed{20}$

(ii)

	Father	Mother	The number of different ways	
⑩	3 females	3 males	${}^5C_3 \times {}^4C_3 = 40$	
	2 females and 1 male	1 female and 2 males	${}^5C_2 \times {}^3C_1 \times {}^3C_1 \times {}^4C_2 = 540$	⑩
	1 female and 2 males	2 females and 1 male	${}^5C_1 \times {}^3C_2 \times {}^3C_2 \times {}^4C_1 = 180$	⑩
	3 males	3 females	${}^3C_3 \times {}^3C_3 = 1$	⑤

Answer = 40 + 540 + 180 + 1 = 761. ⑤

$\boxed{40}$

(*b*) $f(r) - f(r+2) = \dfrac{1}{r(r+2)} - \dfrac{1}{(r+2)(r+4)}$ ⑤

$= \dfrac{(r+4)-r}{r(r+2)(r+4)}$ ⑤

$= 4 \ \dfrac{1}{r(r+2)(r+4)}$

$= 4U_r$ ⑤

$\therefore\ A = 4.$ ⑤

$\boxed{20}$

$\therefore \quad 4U_r \quad = f(r) - f(r+2)$ for $r \in \mathbb{Z}^+$.

$r = 1 \quad : \quad 4U_1 \quad = f(1) - \cancel{f(3)}$

$r = 2 \quad : \quad 4U_2 \quad = f(2) - f(4)$ ⑤

$r = 3 \quad : \quad 4U_3 \quad = \cancel{f(3)} - \cancel{f(5)}$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$r = n-2 \quad : \quad 4U_{n-2} = \cancel{f(n-2)} - \cancel{f(n)}$

$r = n-1 \quad : \quad 4U_{n-1} = \cancel{f(n-1)} - f(n+1)$ ⑤

$r = n \quad : \quad 4U_n \quad = \quad \cancel{f(n)} - f(n+2)$

$$4\sum_{r=1}^{n} U_r = f(1) + f(2) - f(n+1) - f(n+2)$$ ⑩

$$= \frac{1}{3} + \frac{1}{8} - \frac{1}{(n+1)(n+3)} - \frac{1}{(n+2)(n+4)}$$ ⑩

$$\therefore \sum_{r=1}^{n} U_r = \frac{11}{96} - \frac{1}{(n+1)(n+3)} - \frac{1}{(n+2)(n+4)}$$ ⑤

$\boxed{35}$

$$\lim_{n\to\infty}\sum_{r=1}^{n} U_r = \lim_{n\to\infty}\left\{\frac{11}{96}-\frac{1}{4(n+1)(n+3)}-\frac{1}{4(n+2)(n+4)}\right\} \quad ⑤$$

$$=\frac{11}{96}. \quad ⑤$$

$$\therefore \sum_{r=1}^{\infty} U_r \text{ is convergent and its sum } =\frac{11}{96}. \quad ⑤$$

$\boxed{15}$

$$\lim_{n\to\infty}\sum_{r=1}^{n}\left(mU_r + U_{n+1-r}\right)$$

$$=\lim_{n\to\infty}\left(m\sum_{r=1}^{n} U_r + \sum_{r=1}^{n} U_{n+1-r}\right)$$

$$=\lim_{n\to\infty}\left(m\sum_{r=1}^{n} U_r + \sum_{r=1}^{n} U_r\right) \quad ⑩$$

$$=(m \quad 1)+\sum_{r=1}^{\infty} U_r$$

$$\therefore (m+1)\frac{11}{96}=\frac{11}{32} \quad ⑤$$

$$\therefore m+1 = 3 \text{ and } m = 2. \quad ⑤$$

$\boxed{20}$

13.(*a*) Let $a, b \in \mathbb{R}$, $\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 3 & a & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & a & b \\ 3 & b & a \end{pmatrix}$. It is given that $2\mathbf{A} + \mathbf{B} = \begin{pmatrix} 2 & 4 & 3 \\ 9 & 5 & 4 \end{pmatrix}$.

Show that $a = 0$ and $b = 5$.

With these values for a and b, let $\mathbf{C} = \mathbf{A}\mathbf{B}^{\mathrm{T}}$.

Find $\mathbf{C}$ and write down $\mathbf{C}^{-1}$.

Find the matrix $\mathbf{D}$ such that $\mathbf{DC} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

(*b*) Let $z_1, z_2 \in \mathbb{C}$. Show that

(i) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(ii) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

(iii) $z_1 \overline{z_1} = |z_1|^2$

Using the result that $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$ for $z_2 \neq 0$, show that if $|z_1| = 1$ and $z_1 \neq \pm 1$, and also if $\dfrac{z_1 + z_2}{1 + z_1 z_2}$ is real, then $|z_2| = 1$.

(*c*) Express $\sqrt{3} + i$ is the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $0 < \theta < \dfrac{\pi}{2}$.

Using De Moivre's theorem, show that $\dfrac{\left(\sqrt{3}+i\right)^{24}}{2^{23}(1+i)} = 1 \quad \cancel{i}.$

(*a*) $2\mathbf{A} + \mathbf{B} = \begin{pmatrix} 2 & 4 & 3 \\ 9 & 5 & 4 \end{pmatrix}$.

$$2\begin{pmatrix} 1 & 2 & -1 \\ 3 & a & 2 \end{pmatrix} + \begin{pmatrix} 0 & a & b \\ 3 & b & a \end{pmatrix} = \begin{pmatrix} 2 & 4 & 3 \\ 9 & 5 & 4 \end{pmatrix}$$

⑩ $\begin{pmatrix} 2 & 4+a & -2+b \\ 6+3 & 2a+b & 4+a \end{pmatrix} = \begin{pmatrix} 2 & 4 & 3 \\ 9 & 5 & 4 \end{pmatrix}$

$\Leftrightarrow 4 + a = 4$, $-2 + b = 3$ and $2a + b = 5$ ⑩ for any two.

$\Leftrightarrow a = 0$ and $b = 5$. ⑤

$\boxed{25}$

$\mathbf{C} = \mathbf{AB}^{\mathrm{T}}$ ⑤

$$= \begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 0 & 5 \\ 5 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 13 \\ 10 & 9 \end{pmatrix}$$ ⑩

$$\mathbf{C}^{-1} = -\frac{1}{175}\begin{pmatrix} 9 & -13 \\ -10 & -5 \end{pmatrix}.$$ ⑩

$\boxed{25}$

$$\mathbf{DC} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\therefore\ \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{C}^{-1}$$ ⑤

$$= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\left[\frac{1}{175}\begin{pmatrix} 9 & -13 \\ -10 & -5 \end{pmatrix}\right]$$

$$= \frac{1}{175}\begin{pmatrix} 9 & -13 \\ -20 & -10 \end{pmatrix}.$$ ⑩

$\boxed{15}$

(*b*) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ where $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

(i) $\overline{z_1 + z_2} = \overline{(x_1 + x_2) + i(y_1 + y_2)}$

$= (x_1 + x_2) - i(y_1 + y_2)$ ⑤

$= (x_1 - iy_1) + (x2 - iy_2)$

$= \overline{z}_1 \quad \overline{z}_2 +.$ ⑤

$\boxed{10}$

(ii) $\overline{z_1 z_2} = \overline{(x_1 + iy_1) + (x_2 + iy_2)}$

$= \overline{(x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)}$

$= (x_1x_2 - y_1y_2) - i(x_1y_2 + y_1x_2)$ ⑤

$\overline{z}_1 \cdot \overline{z}_2 = (x_1 \ -iy_1)(x_2 \ -iy_2)$

$= (x_1x_2 - y_1y_2) - i(x_1y_2 + y_1x_2)$ ⑤

$\therefore \overline{z_1 z_2} = \overline{z}_1 \overline{z}_2.$ ⑤

$\boxed{15}$

(iii) $z_1\overline{z}_1 = (x_1 \quad iy_1)(x_1 \quad iy_1)$

$= x_1^2 \quad y_1^2 +$ ⑤

$= |z_1|^2 .$ ⑤

$\boxed{10}$

$\overline{\left(\frac{z_1 + z_2}{1 + z_1 z_2}\right)} = \frac{z_1 + z_2}{1 + z_1 z_2}$ ⑤

$\Rightarrow \frac{\overline{z}_1 + \overline{z}_2}{1 + \overline{z}_1\overline{z}_2} = \frac{z_1 + z_2}{1 + z_1 z_2}$ ⑤

$\Rightarrow \overline{z}_1 + \overline{z}_1 z_1 z_2 + \overline{z}_2 + \overline{z}_2 z_1 z_2 = z_1 + z_2 + \overline{z}_1\overline{z}_2 z_1 + \overline{z}_1\overline{z}_2 z_2 = 0$

$\Rightarrow \overline{z}_1 + |\overline{z}_1|^2 z_2 + \overline{z}_2 + z_1|\overline{z}_2|^2 = z_1 + z_2 + \overline{z}_2|\overline{z}_1|^2 + \overline{z}_1|\overline{z}_2|^2 = 0$ ⑤

$\Rightarrow \overline{z}_1 + \cancel{z_2} + \cancel{\overline{z}_2} + z_1|z_2|^2 = z_1 + \cancel{z_2} + \cancel{\overline{z}_2} + \overline{z}_1|z_2|^2 = 0$

$\Rightarrow (z_1 - \overline{z}_1)\left(|\overline{z}_2|^2 - 1\right) = 0$ ⑤

$\Rightarrow |z_2|^2 - 1 = 0 \quad (\because \overline{z}_1 \quad z_1) \quad \neq$

$\Rightarrow |z_2| = 1.$ ⑤

$\left(|z_1| = 1 \text{ and } z_1 \quad 1\right) \quad \neq \pm$

$\boxed{25}$

(c) $\sqrt{3}+i=2\left(\frac{\sqrt{3}}{2}\quad i\frac{1}{2}\right)+$ ⑤

$=2\left(\cos\frac{\pi}{6}\quad i\sin\frac{\pi}{6}\right)+$ ⑤

$r=2$ and $\theta=\frac{\pi}{6}$.

$\frac{\sqrt{3}+i}{2^{23}(1+i)}=\frac{2^{24}(\cos 4\pi+i\sin 4\pi)}{2^{23}(1+i)}$ ⑤

$=\frac{2}{1+i}\quad\frac{1-i}{1-i}\times$ ⑤

$=\frac{2(1-i)}{2}$

$=1-i.$ ⑤

$\boxed{25}$

---

14. (*a*) Let $f(x)=\dfrac{px+q}{(x-1)(x-2)}$ for $x \in \mathbb{R}-\{1, 2\}$, where $p, q \in \mathbb{R}$. It is given that the graph of $y=f(x)$ has a stationary point at $(0, 1)$. Show that $p=-3$ and $q=2$.

For these values of p and q, show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x)=\dfrac{x(3x-4)}{(x-1)^2(x-2)^2}$ for $x \neq 1, 2$, and find the intervals on which $f(x)$ is decreasing and the intervals on which $f(x)$ is increasing.

Sketch the graph of $y=f(x)$ indicating the asymptotes and the turning points.

Hence, find the number of real solutions to the equation $x^2(x-1)(x-2)=2-3x$.

(*b*) A cylinder with a top and a bottom is made to have a volume of 1024π cm^3. Let r cm be the radius of the cylinder. Show that the total surface area S cm^2 of the cylinder is given by $S=2\pi\left(\dfrac{1024}{r}+r^2\right)$ for $r>0$.

Show that S is minimum when $r=8$.

(*a*) Since $f(0)=1$, we have $\dfrac{q}{2}=1$.

$\therefore\ q=2$ ⑤

$$f'(x)=\frac{(x-1)(x-2)p-(px+q)(x-1+x-2)}{(x-1)^2(x-2)^2} \quad ⑩ \text{ for } x \neq 1, 2.$$

Since $f'(0)=0$, we have $2p-q(-3)=0$. ⑤

$\therefore\ 2p=-3q$

$\quad\ =-6$

$\therefore\ p=-3$ ⑤

$\boxed{25}$

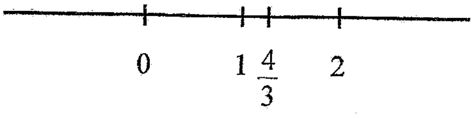
$$f'(x)=\frac{-3(x^2-3x+2)-(-3x+2)(2x-3)}{(x-1)^2(x-2)^2} \quad ⑤$$

$$=\frac{-3x^2+9x-6+6x^2-13x+6}{(x-1)^2(x-2)^2}$$

$$=\frac{3x^2-4x}{(x-1)^2(x-2)^2}$$

$$=\frac{x(3x-4)}{(x-1)^2(x-2)^2} \quad ⑤ \text{ for } x \neq 1, 2.$$

$f'(x)=0 \Leftrightarrow x=0$ or $x=\dfrac{4}{3}$. ⑤



	$-\infty < x < 0$	$0 < x < 1$	$1 < x < \frac{4}{3}$	$\frac{4}{3} < x < 2$	$2 < x < \infty$
Sign of $f'(x)=\dfrac{x(3x-4)}{(x-1)^2(x-2)^2}$	$\frac{(-)(-)}{(+)}$ $=(+)$ ⑤	$\frac{(+)(-)}{(+)}$ $=(-)$ ⑤	$\frac{(+)(-)}{(+)}$ $=(-)$ ⑤	$\frac{(+)(+)}{(+)}$ $=(+)$ ⑤	$\frac{(+)(+)}{(+)}$ $=(+)$ ⑤
$f(x)$	↗ ∩	↘	↘ ∪	↗	↗

⑩

Increasing on : $(-\infty, 0]$, $[\frac{4}{3}, 2)$ and $(2, \infty)$

Decreasing on : $(0, 1]$ and $[1, \frac{4}{3})$

$\boxed{50}$

Local maximum at $(0, 1)$ ⑤

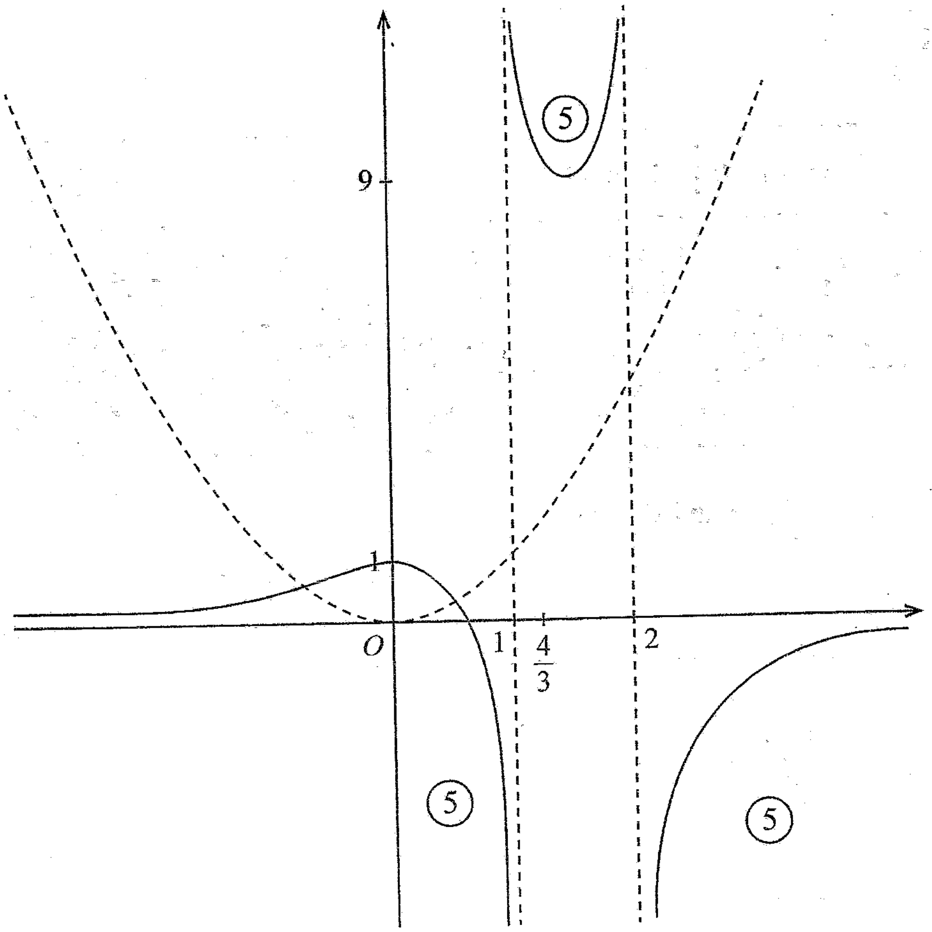
Local minimum at $(\frac{4}{3}, 9)$ ⑤

$\lim\limits_{x\to 1-} f(x) = -\infty$, $\lim\limits_{x\to 1+} f(x) = \infty$

⑤

$\lim\limits_{x\to 2-} f(x) = \infty$, $\lim\limits_{x\to 2+} f(x) = -\infty$

$\lim\limits_{x\to \pm\infty} f(x) = 0$. ⑤



$\boxed{35}$

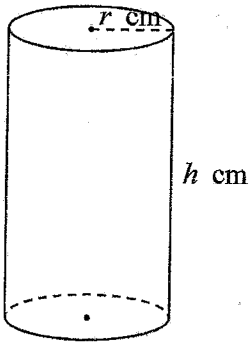
$x^2(x-1)(x-2) = 2 - 3x$

$\Leftrightarrow x^2 = \dfrac{2-3x}{(x-1)(x-2)}$ ⑤

$\Leftrightarrow x^2 = f(x)$

$\therefore$ The number of real solutions = 2. ⑤ $\boxed{10}$

(*b*)



$$S = 2\pi rh \quad 2\pi r^2 +$$

$$\pi r^2 h = 1024\pi$$

$$\therefore \quad h = \frac{1024}{r^2} \quad ⑤$$

$$\therefore \quad S = 2\pi r \ \frac{1024}{r^2} + \ 2\pi r^2$$

$$= 2\pi\left(\frac{1024}{r} \quad r^2\right) \quad ⑤$$

$\boxed{10}$

$$\frac{dS}{dr} = 2\pi\left(-\ \frac{1024}{r^2} + \ 2r\right) \quad ⑤$$

$$\frac{dS}{dr} = 0 < ⑤ \Rightarrow \ \frac{1024}{r^2} = \ 2r$$

$$\Leftrightarrow r^3 = 512$$

$$\Leftrightarrow r = 8\,. \quad ⑤$$

$\frac{dS}{dr} < 0$ for $0 < r < 8$.

$\frac{dS}{dr} > 0$ for $r > 8$.

$\therefore$ S is minimum when $r = 8$. ⑤

$\boxed{20}$

15. (*a*) Find the values of the real constants A and B such that $3t^2+4 = A(t^2-2t+4)+Bt(t+1)$ for all $t \in \mathbb{R}$.

Hence or otherwise, find $\int \frac{3t^2+4}{(t+1)(t^2-2t+4)}\,dt$.

(*b*) Using the substitution $u = x+\sqrt{x^2+3}$, show that $\int_0^1 \frac{1}{\sqrt{x^2+3}}\,dx = \frac{1}{2}\ln 3$.

Let $J = \int_0^1 \sqrt{x^2+3}\,dx$. Using integration by parts, show that $2J = 2 + \int_0^1 \frac{3}{\sqrt{x^2+3}}\,dx$.

Deduce that $J = 1+\frac{3}{4}\ln 3$.

(*c*) Using the formula $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, where a is a constant, show that

$$\int_0^{\frac{\pi}{4}} \ln\left(\frac{\cos x}{\cos x+\sin x}\right)dx = \frac{\pi}{8}\ln\left(\frac{1}{2}\right).$$

(*a*) $3t^2+4 = A(t^2-2t+4)+Bt(t+1)$

$= (A+B)t^2+(-2A+B)t+4A$ ⑤

Coefficients of t^2 : $3 = A+B$

Coefficients of t^1 : $0 = -2A+B$

Coefficients of t^0 : $4 = 4A$

$\therefore$ $A = 1$ and $B = 2$.
⑤ ⑤

$\boxed{15}$

$$\therefore \quad \frac{3t^2+4}{(t+1)(t^2-2t+4)} = \frac{1}{t+1} \quad \frac{2t}{t^2-2t+4}. \qquad ⑩$$

$$\int \frac{3t^2+4}{(t+1)(t^2-2t+4)}\,\mathrm{d}t = \int \left\{ \frac{1}{t+1} \quad \frac{2t-2+2}{t^2-2t+4} \right\} \mathrm{d}t$$

$$= \int \frac{1}{t+1}\mathrm{d}t \quad \int \frac{2t-2}{t^2-2t+4}\mathrm{d}t \quad 2\int \frac{1}{(t-1)^2+3}\mathrm{d}t$$

$$= \ln|t \quad 1| \quad \ln|t^2 \quad 2t \quad 4| \quad \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t-1}{\sqrt{3}}\right) + C, \qquad ⑤$$

⑤ ⑤ ⑤

where C is an arbitrary constant.

$\boxed{30}$

(*b*) $u = x + \sqrt{x^2+3}$.

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 1 \quad \frac{x}{\sqrt{x^2+3}} \quad \frac{u}{\sqrt{x^2+3}} \qquad ⑤$$

$$\therefore \quad \frac{1}{\sqrt{x^2+3}}\mathrm{d}x = \frac{1}{u}\mathrm{d}u$$

$x = 0 \Rightarrow u = \sqrt{3}$.

$x = 1 \Rightarrow u = 3$. ⑤

$$\text{So,} \int_0^1 \frac{1}{\sqrt{x^2+3}}\mathrm{d}x = \int_{\sqrt{3}}^{3} \frac{1}{u}\mathrm{d}u = \ln|u|\Big|_{\sqrt{3}}^{3} = \ln 3 \quad \ln\sqrt{3}$$

⑤ ⑤

$$= \ln\sqrt{3} \quad \frac{1}{2}\ln 3. \qquad ⑤$$

$\boxed{25}$

$$J = \int_0^1 \sqrt{x^2+3}\,\mathrm{d}x = x\sqrt{x^2+3}\Big|_0^1 - \int_0^1 \frac{x^2}{\sqrt{x^2+3}}\mathrm{d}x \qquad ⑩$$

$$= 2 \quad \int_0^1 \frac{x^2+3-3}{\sqrt{x^2+3}}\mathrm{d}x \qquad ⑤$$

$$\therefore \quad \frac{3t^2+4}{(t+1)(t^2-2t+4)} = \frac{1}{t+1} \quad \frac{2t}{t^2-2t+4}. \quad ⑩$$

$$\int \frac{3t^2+4}{(t+1)(t^2-2t+4)}\,\mathrm{d}t = \int \left\{ \frac{1}{t+1} \quad \frac{2t-2+2}{t^2-2t+4} \right\} \mathrm{d}t$$

$$= \int \frac{1}{t+1}\mathrm{d}t \quad \int \frac{2t-2}{t^2-2t+4}\mathrm{d}t \quad 2\int \frac{1}{(t-1)^2+3}\mathrm{d}t$$

$$= \ln|t \quad 1| \,.\, \ln|t^2 \quad 2t \quad 4| \quad \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{t-1}{\sqrt{3}}\right) + C, \quad ⑤$$

⑤ ⑤ ⑤

where C is an arbitrary constant.

$\boxed{30}$

(*b*) $u = x + \sqrt{x^2+3}$.

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 1 \quad \frac{x}{\sqrt{x^2+3}} \quad \frac{u}{\sqrt{x^2+3}} \quad ⑤$$

$$\therefore \quad \frac{1}{\sqrt{x^2+3}}\mathrm{d}x = \frac{1}{u}\mathrm{d}u$$

$x = 0 \Rightarrow u = \sqrt{3}$.

$x = 1 \Rightarrow u = 3$. ⑤

$$\text{So,} \quad \int_0^1 \frac{1}{\sqrt{x^2+3}}\mathrm{d}x = \int_{\sqrt{3}}^{3} \frac{1}{u}\mathrm{d}u = \ln|u|\Big|_{\sqrt{3}}^{3} = \ln 3 \quad \ln\sqrt{3}$$

⑤ ⑤

$$= \ln\sqrt{3} \quad \frac{1}{2}\ln 3. \quad ⑤$$

$\boxed{25}$

$$J = \int_0^1 \sqrt{x^2+3}\,\mathrm{d}x = x\sqrt{x^2+3}\Big|_0^1 - \int_0^1 \frac{x^2}{\sqrt{x^2+3}}\mathrm{d}x \quad ⑩$$

$$= 2 \quad \int_0^1 \frac{x^2+3-3}{\sqrt{x^2+3}}\mathrm{d}x \quad ⑤$$

$$- \therefore \quad J = 2 \quad \int_0^1 +\sqrt{x^2+3}\,\mathrm{d}x \quad \int_0^1 \frac{3}{\sqrt{x^2+3}}\,\mathrm{d}x \qquad ⑩$$

$$\Rightarrow 2J = 2 \quad \int_0^1 \frac{3}{\sqrt{x^2+3}}\,\mathrm{d}x. \qquad ⑤$$

$\boxed{30}$

$$\therefore \quad J = 1+ \ \frac{3}{2}\ \frac{1}{2}\ln 2$$

$$= 1 \ +\frac{3}{4}\ln 2 \qquad ⑩$$

$\boxed{10}$

(c) Let $I = \int_0^{\frac{\pi}{4}} \ln\left(\frac{\cos x}{\cos x + \sin x}\right)\mathrm{d}x$.

$$\text{Then } \ I = \int_0^{\frac{\pi}{4}} \ln\left(\frac{\cos\left(\frac{\pi}{4} - x\right)}{\cos\left(\frac{\pi}{4} - x\right) + \sin\left(\frac{\pi}{4} - x\right)}\right)\mathrm{d}x \qquad ⑤$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{\cos\frac{\pi}{4}\cos x + \sin\frac{\pi}{4}\sin x}{\cos\frac{\pi}{4}\cos x + \sin\frac{\pi}{4}\sin x + \sin\frac{\pi}{4}\cos x - \cos\frac{\pi}{4}\sin x}\right)\mathrm{d}x \qquad ⑩$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{\cos x + \sin x}{2\cos x}\right)\mathrm{d}x \qquad ⑤$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{1}{2}\right)\mathrm{d}x \quad \int_0^{\frac{\pi}{4}} \ln\left(\frac{\cos x}{\cos x + \sin x}\right)\mathrm{d}x \qquad ⑩$$

$$\therefore \quad = I \quad \ln\left(\frac{1}{2}\right)x\Big|_0^{\frac{\pi}{4}} \quad I. - \qquad ⑤$$

$$\Rightarrow 2I = \ln\left(\frac{1}{2}\right)\ \frac{\pi}{4}$$

$$\therefore \quad I = \frac{\pi}{8}\ln\left(\frac{1}{2}\right). \qquad ⑤$$

$\boxed{40}$

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16. Let $A \equiv (1, 2)$ and $B \equiv (a, b)$, where $a, b \in \mathbb{R}$. It is given that the perpendicular bisector l of the line segment AB has the equation $x + y - 4 = 0$. Find the values of a and b.

Let $C \equiv (3, 1)$. Show that the point C lies on the line l and find $A\hat{C}B$.

Let S be the circle through the points A, B and C. Show that the centre of S is given by $\left(\frac{13}{6}, \frac{11}{6}\right)$ and find the equation of S.

Hence, find the equation of the circle passing through the points A, B and the point $D \equiv (0, 3)$.

The mid-point of $AB \equiv \left(\frac{1+a}{2}, \frac{2+b}{2}\right)$. ⑤

This point lies on l.

$\therefore \frac{1+a}{2} + \frac{2+b}{2} - 4 = 0$

$\therefore a + b = 5$ ——— ① ⑤

Also, l is perpendicular to AB.

⑤

$\therefore \left(\frac{b-2}{a-1}\right) \times (-1) = -1$. ⑤

$\therefore b - 2 = a - 1$

$\therefore b - a = 1$ ——— ② ⑤

Now, ① and ② give us $a = 2$ and $b = 3$.
⑤ ⑤

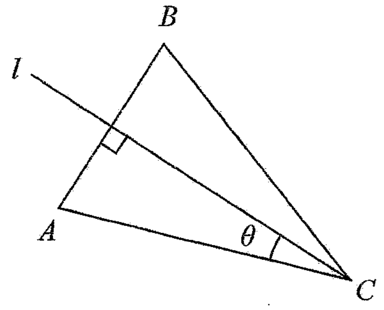
So, $A \equiv (1, 2)$, $B \equiv (2, 3)$ and $C \equiv (3, 1)$. $\boxed{35}$

Substitute $x = 3$ and $y = 1$ in $x + y - 4 = 0$. ⑤

L.H.S. $= 3 + 4 - 1$

$= 0 =$ R.H.S.

$\therefore C$ lies on l. ⑤



Let the angle between AC and l be θ.

Then θ is acute.

The slope of $AC = \frac{1}{2}$. ⑤

$$\tan\theta = \left| \frac{-\frac{1}{2} - (-1)}{1 + \left(-\frac{1}{2}\right) \times (-1)} \right| = \frac{1}{3}$$ ⑩

$$\theta = \tan^{-1}\left(\frac{1}{3}\right).$$ ⑤

$$\therefore \quad A\hat{C}B = 2\theta \quad 2\tan^{-1}\left(\frac{1}{3}\right).$$ ⑤

$\boxed{35}$

Let m be the perpendicular bisector of AC.

The slope of $m = 2$. ⑤

The mid-point of $AC \equiv \left(2, \frac{3}{2}\right)$. ⑤

The equation of m: $y - \frac{3}{2} = 2(x \quad 2)$ ⑤

i.e. $y - 2x + \frac{5}{2} = 0$.

The centre of S is the point of intersection of l and m. ⑤

Solving $x + y - 4 = 0$ and $y - 2x + \frac{5}{2} = 0$ simultaneously, ⑤ we get

$$3x = 4 \quad \frac{5}{2}.$$

$$\therefore \quad x = \frac{16}{3} \text{ and } y = 4 \quad \frac{13}{6} \quad \frac{11}{6}.$$

⑤ ⑤

$$\therefore \text{ The centre of } S \equiv \left(\frac{13}{6}, \frac{11}{6}\right).$$

The radius of $S = \sqrt{\left(\frac{13}{6} - 1\right)^2 + \left(\frac{11}{6} - 2\right)^2}$ ⑩

$$= \sqrt{\frac{49}{36} + \frac{1}{36}}$$

$$= \sqrt{\frac{25}{18}}$$ ⑤

$\therefore$ The equation of S: $\left(x-\frac{13}{6}\right)^2+\left(y-\frac{11}{6}\right)^2=\frac{25}{18}$. ⑩

$x^2+y^2-\frac{13}{3}x-\frac{11}{3}y+\frac{20}{3}=0$

$\boxed{60}$

$\therefore$ The equation of AB: $y-2=1(x-1)$ ⑤

i.e. $x-y+1=0$

The equation of any circle through A and B is given by $\left(x-\frac{13}{6}\right)^2+\left(y-\frac{11}{6}\right)^2-\frac{25}{18}+\lambda(x-y+1)=0$, where $\lambda \in \mathbb{R}$. ⑩

For this to go through $D\equiv(0,3)$, we must have $\left(\frac{13}{6}\right)^2+\left(\frac{7}{6}\right)^2-\frac{25}{18}+\lambda(-2)=0$.

$\lambda=\frac{21}{9}\quad\frac{7}{3}\underset{\cdot}{=}$ ⑤

$\therefore$ The required equation is $x^2+y^2-2x-6y+9=0$.

$\boxed{20}$

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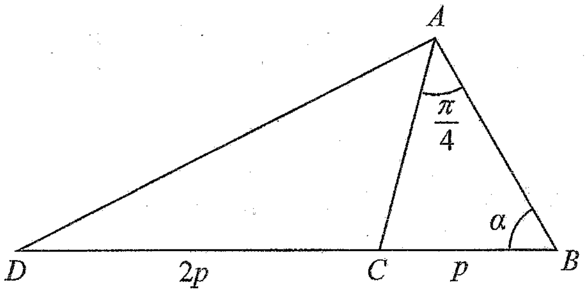
17. (*a*) Express $6\cos 2x - 8\sin 2x$ in the form $R\cos(2x+\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

Hence, solve $6\cos 2x - 8\sin 2x = 5$.

Express $24\cos^2 x - 32\sin x\cos x$ in the form $a\cos 2x + b\sin 2x + c$, where a, b, c $(\in \mathbb{R})$ are constants to be determined.

Deduce the minimum value of $24\cos^2 x - 32\sin x\cos x$.

(*b*)



In the triangle ABC shown in the figure, $BC = p$, $B\hat{A}C = \frac{\pi}{4}$ and $A\hat{B}C = \alpha$. The point D lies on the extended line BC such that $CD = 2p$.

Show that $AB = p(\cos\alpha + \sin\alpha)$.

Find AD^2 in terms of p and α.

Deduce that if $AD = 3p$, then $\alpha = \tan^{-1}(5)$.

(*c*) Solve the equation $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$.

(*a*) $6\cos 2x - 8\sin 2x$ $\qquad \sqrt{6^2+8^2} = 10$

$$= 10\left(\frac{6}{10}\cos 2x - \frac{8}{10}\sin 2x\right) \quad ⑤$$

$$= 10\left(\frac{3}{5}\cos 2x - \frac{4}{5}\sin 2x\right)$$

$= 10(\cos\alpha\cos 2x - \sin\alpha\sin 2x)$, where $0 < \alpha < \frac{\pi}{2}$ is such that $\cos\alpha = \frac{3}{5}$ and $\sin\alpha = \frac{4}{5}$. ⑤
⑤

$= 10\cos(2x \quad \alpha)$. ⑤

$\therefore$ $R = 10$ and $\alpha = \tan^{-1}\left(\frac{4}{3}\right)$. ⑤

$\boxed{25}$

$6\cos 2x - 8\sin 2x = 5$

$10\cos(2x+\alpha) = 5$ ⑤

$\therefore \; \cos(2x+\alpha) = \frac{1}{2} = \cos\frac{\pi}{3}$ ⑤

$2x+\alpha = 2n\pi \; \pm\frac{\pi}{3}$, where $n \in \mathbb{Z}$. ⑤

$\therefore \; x = n\pi \; \frac{\alpha}{2} \pm \frac{\pi}{6}$, where $n \in \mathbb{Z}$. ⑤

$\boxed{20}$

$24\cos^2 x - 32\sin x\cos x$

$= 24\left(\frac{\cos 2x + 1}{2}\right) - 32 \times \frac{1}{2}\sin 2x$ ⑩

$= 12\cos 2x - 16\sin 2x + 12$

$\therefore \; a = 12, \quad b = -16$ and $c = 12$. ⑤

$\boxed{15}$

$24\cos^2 x - 32\sin x\cos x$

$= 12\cos 2x - 16\sin 2x + 12$

$= 2(6\cos 2x - 8\sin 2x) + 12$

$= 20\,\underbrace{\cos(2x+\alpha)}_{\min\,=\,-1} + 12$ ⑤

$\therefore$ The required minimum $= -20 + 12$

$= -8.$ ⑤

$\boxed{10}$

(*b*) Sine Rule for the $\triangle ABC$:

$$\frac{AB}{\sin\left(\frac{\pi}{4}+\alpha\right)}=\frac{p}{\sin\frac{\pi}{4}} \quad ⑩$$

$$\therefore \quad AB=\sqrt{2}p\left(\sin\frac{\pi}{4}\cos\alpha+\cos\frac{\pi}{4}\sin\alpha\right) \quad ⑤$$

$$=\sqrt{2}p\left(\frac{1}{\sqrt{2}}\cos\alpha+\frac{1}{\sqrt{2}}\sin\alpha\right)$$

$$=p(\cos\alpha \quad \sin\alpha). \quad ⑤$$

$\boxed{20}$

Cosine Rule for the $\triangle ABD$:

$$AD^2=AB^2+BD^2-2AB\cdot BD\cos\alpha \quad ⑤$$

$$=p^2(\cos\alpha \quad \sin\alpha)^2 \quad 9p^2+ \quad 2p(\cos\alpha \quad \sin\alpha)\ 3p\cos\alpha \quad ⑤$$

Suppose that $AD = 3p$.

Then $9p^2=p^2(\cos\alpha \quad \sin\alpha)^2 \quad 9p^2+ \quad 6p^2(\cos\alpha \quad \sin\alpha)\ \cos\alpha$ ⑤

$$\therefore \quad (\cos\alpha+\sin\alpha)^2-6(\cos\alpha+\sin\alpha)\cos\alpha=0$$

$\Rightarrow \quad \cos\alpha+\sin\alpha=6\cos\alpha$ ⑤ $\left(0<\alpha<\frac{\pi}{2}\right)$

$\Rightarrow \quad \sin\alpha=5\cos\alpha$

$\Rightarrow \quad \tan\alpha=5$

$\Rightarrow \quad \alpha=\tan^{-1}(5).$ ⑤

$\boxed{25}$

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12. (*a*) The parents of a family decide to invite 6 out of 15 of their close relatives for a dinner. While the father has 5 close female relatives and 3 close male relatives, the mother has 3 close female relatives and 4 close male relatives.

Find the number of different ways in which

(i) the father can invite 3 of his close female relatives and the mother can invite 3 of her close male relatives,

(ii) the father can invite 3 of his close relatives and the mother can invite 3 of her close relatives so that 3 males and 3 females are invited.

(*b*) Let $U_r = \dfrac{1}{r(r+2)(r+4)}$ and $f(r) = \dfrac{1}{r(r+2)}$ for $r \in \mathbb{Z}^+$.

Determine the value of the real constant A such that $f(r) - f(r+2) = AU_r$ for $r \in \mathbb{Z}^+$.

Hence, show that $\sum\limits_{r=1}^{n} U_r = \dfrac{11}{96} - \dfrac{1}{4(n+1)(n+3)} - \dfrac{1}{4(n+2)(n+4)}$ for $n \in \mathbb{Z}^+$.

Show further that the infinite series $\sum\limits_{r=1}^{\infty} U_r$ is convergent and find its sum.

Find the value of the real constant m such that $\lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} (mU_r + U_{n+1-r}) = \dfrac{11}{32}$.

12. (*a*)

Father		Mother	
Female relatives	Male relatives	Female relatives	Male relatives
5	3	3	4

(i) The number of different ways, the father can invite 3 close female relatives $= {}^5C_3$ ⑤

The number of different ways, the mother can invite 3 close male relatives $= {}^4C_3$ ⑤

∴ The number of different ways, the father can invite 3 close female relatives and the mother can invite 3 close male relatives $= {}^5C_3 \times {}^4C_3$ ⑤

$$= \frac{5!}{3!2!} \times \frac{4!}{3!1!} = \frac{5 \times \overset{2}{\cancel{4}}}{\cancel{2}} \quad 4 \times 40.$$ ⑤

$\boxed{20}$

(ii)

	Father	Mother	The number of different ways	
⑩	3 females	3 males	${}^5C_3 \times {}^4C_3 = 40$	
	2 females and 1 male	1 female and 2 males	${}^5C_2 \times {}^3C_1 \times {}^3C_1 \times {}^4C_2 = 540$	⑩
	1 female and 2 males	2 females and 1 male	${}^5C_1 \times {}^3C_2 \times {}^3C_2 \times {}^4C_1 = 180$	⑩
	3 males	3 females	${}^3C_3 \times {}^3C_3 = 1$	⑤

Answer = 40 + 540 + 180 + 1 = 761. ⑤

$\boxed{40}$

(*b*) $f(r) - f(r+2) = \dfrac{1}{r(r+2)} - \dfrac{1}{(r+2)(r+4)}$ ⑤

$= \dfrac{(r+4) - r}{r(r+2)(r+4)}$ ⑤

$= 4 \dfrac{1}{r(r+2)(r+4)}$

$= 4U_r$ ⑤

$\therefore A = 4.$ ⑤

$\boxed{20}$

$\therefore \quad 4U_r = f(r) - f(r+2)$ for $r \in \mathbb{Z}^+$.

$r = 1 \quad : \quad 4U_1 = f(1) - f(\cancel{3})$

$r = 2 \quad : \quad 4U_2 = f(2) - f(4)$ ⑤

$r = 3 \quad : \quad 4U_3 = f(\cancel{3}) - f(\cancel{5})$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$r = n-2 \quad : \quad 4U_{n-2} = f(\cancel{n-2}) - f(\cancel{n})$

$r = n-1 \quad : \quad 4U_{n-1} = f(\cancel{n-1}) - f(n+1)$ ⑤

$r = n \quad : \quad 4U_n = f(\cancel{n}) - f(n+2)$

$4\sum_{r=1}^{n} U_r = f(1) + f(2) - f(n+1) - f(n+2)$ ⑩

$= \dfrac{1}{3} + \dfrac{1}{8} - \dfrac{1}{(n+1)(n+3)} - \dfrac{1}{(n+2)(n+4)}$ ⑩

$\therefore \sum_{r=1}^{n} U_r = \dfrac{11}{96} - \dfrac{1}{(n+1)(n+3)} - \dfrac{1}{(n+2)(n+4)}$ ⑤

$\boxed{35}$

$$\lim_{n\to\infty}\sum_{r=1}^{n}U_r=\lim_{n\to\infty}\left\{\frac{11}{96}-\frac{1}{4(n+1)(n+3)}-\frac{1}{4(n+2)(n+4)}\right\}$$ ⑤

$$=\frac{11}{96}.$$ ⑤

$\therefore\ \sum_{r=1}^{\infty}U_r$ is convergent and its sum $=\frac{11}{96}$. ⑤

$\boxed{15}$

$$\lim_{n\to\infty}\sum_{r=1}^{n}\left(mU_r+U_{n+1-r}\right)$$

$$=\lim_{n\to\infty}\left(m\sum_{r=1}^{n}U_r+\sum_{r=1}^{n}U_{n+1-r}\right)$$

$$=\lim_{n\to\infty}\left(m\sum_{r=1}^{n}U_r+\sum_{r=1}^{n}U_r\right)$$ ⑩

$$=(m\quad 1)\sum_{r=1}^{\infty}U_r$$

$$\therefore (m+1)\frac{11}{96}=\frac{11}{32}$$ ⑤

$\therefore\ m+1=3$ and $m=2$. ⑤

$\boxed{20}$

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13. (*a*) Let $a, b \in \mathbb{R}$, $\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 3 & a & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & a & b \\ 3 & b & a \end{pmatrix}$. It is given that $2\mathbf{A} + \mathbf{B} = \begin{pmatrix} 2 & 4 & 3 \\ 9 & 5 & 4 \end{pmatrix}$.

Show that $a = 0$ and $b = 5$.

With these values for a and b, let $\mathbf{C} = \mathbf{AB}^{\mathrm{T}}$.

Find $\mathbf{C}$ and write down $\mathbf{C}^{-1}$.

Find the matrix $\mathbf{D}$ such that $\mathbf{DC} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

(*b*) Let $z_1, z_2 \in \mathbb{C}$. Show that

(i) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(ii) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

(iii) $z_1 \overline{z_1} = |z_1|^2$

Using the result that $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$ for $z_2 \neq 0$, show that if $|z_1| = 1$ and $z_1 \neq \pm 1$, and also if $\dfrac{z_1 + z_2}{1 + z_1 z_2}$ is real, then $|z_2| = 1$.

(*c*) Express $\sqrt{3} + i$ is the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $0 < \theta < \dfrac{\pi}{2}$.

Using De Moivre's theorem, show that $\dfrac{\left(\sqrt{3} + i\right)^{24}}{2^{23}(1 + i)} = 1 - i$.

(*a*) $2\mathbf{A} + \mathbf{B} = \begin{pmatrix} 2 & 4 & 3 \\ 9 & 5 & 4 \end{pmatrix}$.

$$2\begin{pmatrix} 1 & 2 & -1 \\ 3 & a & 2 \end{pmatrix} + \begin{pmatrix} 0 & a & b \\ 3 & b & a \end{pmatrix} = \begin{pmatrix} 2 & 4 & 3 \\ 9 & 5 & 4 \end{pmatrix}$$

⑩ $\begin{pmatrix} 2 & 4 + a & -2 + b \\ 6 + 3 & 2a + b & 4 + a \end{pmatrix} = \begin{pmatrix} 2 & 4 & 3 \\ 9 & 5 & 4 \end{pmatrix}$

$\Leftrightarrow$ $4 + a = 4$, $-2 + b = 3$ and $2a + b = 5$ ⑩ for any two.

$\Leftrightarrow$ $a = 0$ and $b = 5$. ⑤

$\boxed{25}$

$\mathbf{C} = \mathbf{AB}^{\mathrm{T}}$ ⑤

$$= \begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 0 & 5 \\ 5 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 13 \\ 10 & 9 \end{pmatrix}$$ ⑩

$$\mathbf{C}^{-1} = -\frac{1}{175} \begin{pmatrix} 9 & -13 \\ -10 & -5 \end{pmatrix}.$$ ⑩

$\boxed{25}$

$$\mathbf{DC} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\therefore\ \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{C}^{-1}$$ ⑤

$$= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \left[\frac{1}{175} \begin{pmatrix} 9 & -13 \\ -10 & -5 \end{pmatrix} \right]$$

$$= \frac{1}{175} \begin{pmatrix} 9 & -13 \\ -20 & -10 \end{pmatrix}.$$ ⑩

$\boxed{15}$

(*b*) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ where $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

(i) $\overline{z_1 + z_2} = \overline{(x_1 + x_2) + i(y_1 + y_2)}$

$= (x_1 + x_2) - i(y_1 + y_2)$ ⑤

$= (x_1 - iy_1) + (x2 - iy_2)$

$= \overline{z}_1 \quad \overline{z}_2 + .$ ⑤

$\boxed{10}$

(ii) $\overline{z_1 z_2} = \overline{(x_1 + iy_1) + (x_2 + iy_2)}$

$= \overline{(x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)}$

$= (x_1x_2 - y_1y_2) - i(x_1y_2 + y_1x_2)$ ⑤

$\overline{z}_1 \cdot \overline{z}_2 = (x_1 \;\; -iy_1)(x_2 \;\; -iy_2)$

$= (x_1x_2 - y_1y_2) - i(x_1y_2 + y_1x_2)$ ⑤

$\therefore \overline{z_1 z_2} = \overline{z}_1 \overline{z}_2 .$ ⑤

$\boxed{15}$

(iii) $z_1 \overline{z}_1 = (+x_1 \quad iy_1)(-x_1 \quad iy_1)$

$= x_1^2 \quad y_1^2 +$ ⑤

$= |z_1|^2 .$ ⑤

$\boxed{10}$

$\overline{\left(\frac{z_1 + z_2}{1 + z_1 z_2}\right)} = \frac{z_1 + z_2}{1 + z_1 z_2}$ ⑤

$\Rightarrow \frac{\overline{z}_1 + \overline{z}_2}{1 + \overline{z}_1 \overline{z}_2} = \frac{z_1 + z_2}{1 + z_1 z_2}$ ⑤

$\Rightarrow \overline{z}_1 + \overline{z}_1 z_1 z_2 + \overline{z}_2 + \overline{z}_2 z_1 z_2 = z_1 + z_2 + \overline{z}_1 \overline{z}_2 z_1 + \overline{z}_1 \overline{z}_2 z_2 = 0$

$\Rightarrow \overline{z}_1 + |\overline{z}_1|^2 z_2 + \overline{z}_2 + z_1 |\overline{z}_2|^2 = z_1 + z_2 + \overline{z}_2 |\overline{z}_1|^2 + \overline{z}_1 |\overline{z}_2|^2 = 0$ ⑤

$\Rightarrow \overline{z}_1 + \cancel{z_2} + \cancel{\overline{z}_2} + z_1 |z_2|^2 = z_1 + \cancel{z_2} + \cancel{\overline{z}_2} + \overline{z}_1 |z_2|^2 = 0$

$\Rightarrow (z_1 - \overline{z}_1)\left(|\overline{z}_2|^2 - 1\right) = 0$ ⑤

$\Rightarrow |z_2|^2 - 1 = 0 \quad (\because \overline{z}_1 \quad z_1) \qquad \neq$

$\Rightarrow |z_2| = 1.$ ⑤

$\left(|z_1| = 1 \text{ and } z_1 \qquad 1\right) \quad \neq \pm$

$\boxed{25}$

(*c*) $\sqrt{3}+i=2\left(\frac{\sqrt{3}}{2}\quad i\frac{1}{2}\right)+$ ⑤

$$=2\left(\cos\frac{\pi}{6}\quad i\sin\frac{\pi}{6}\right)+ \quad ⑤$$

$r=2$ and $\theta=\frac{\pi}{6}$.

$$\frac{\sqrt{3}+i}{2^{23}(1+i)}=\frac{2^{24}(\cos 4\pi+i\sin 4\pi)}{2^{23}(1+i)} \quad ⑤$$

$$=\frac{2}{1+i}\quad\frac{1-i}{1-i}\times \quad ⑤$$

$$=\frac{2(1-i)}{2}$$

$$=1-i. \quad ⑤$$

$\boxed{25}$

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14. (*a*) Let $f(x)=\dfrac{px+q}{(x-1)(x-2)}$ for $x \in \mathbb{R}-\{1, 2\}$, where $p, q \in \mathbb{R}$. It is given that the graph of $y=f(x)$ has a stationary point at $(0, 1)$. Show that $p=-3$ and $q=2$.

For these values of p and q, show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x)=\dfrac{x(3x-4)}{(x-1)^2(x-2)^2}$ for $x \neq 1, 2$, and find the intervals on which $f(x)$ is decreasing and the intervals on which $f(x)$ is increasing.

Sketch the graph of $y=f(x)$ indicating the asymptotes and the turning points.

Hence, find the number of real solutions to the equation $x^2(x-1)(x-2)=2-3x$.

(*b*) A cylinder with a top and a bottom is made to have a volume of 1024π cm^3. Let r cm be the radius of the cylinder. Show that the total surface area S cm^2 of the cylinder is given by $S=2\pi\left(\dfrac{1024}{r}+r^2\right)$ for $r>0$.

Show that S is minimum when $r=8$.

(*a*) Since $f(0)=1$, we have $\dfrac{q}{2}=1$.

$\therefore\ q=2$ ⑤

$f'(x)=\dfrac{(x-1)(x-2)p-(px+q)(x-1+x-2)}{(x-1)^2(x-2)^2}$ ⑩ for $x \neq 1, 2$.

Since $f'(0)=0$, we have $2p-q(-3)=0$. ⑤

$\therefore\ 2p=-3q$

$=-6$

$\therefore\ p=-3$ ⑤

$\boxed{25}$

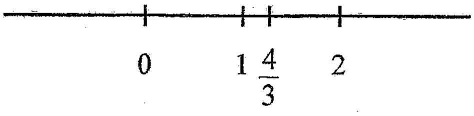
$f'(x)=\dfrac{-3(x^2-3x+2)-(-3x+2)(2x-3)}{(x-1)^2(x-2)^2}$ ⑤

$=\dfrac{-3x^2+9x-6+6x^2-13x+6}{(x-1)^2(x-2)^2}$

$=\dfrac{3x^2-4x}{(x-1)^2(x-2)^2}$

$=\dfrac{x(3x-4)}{(x-1)^2(x-2)^2}$ ⑤ for $x \neq 1, 2$.

$f'(x)=0 \Leftrightarrow x=0$ or $x=\dfrac{4}{3}$. ⑤



	$-\infty<x<0$	$0<x<1$	$1<x<\frac{4}{3}$	$\frac{4}{3}<x<2$	$2<x<\infty$
Sign of $f'(x)=\dfrac{x(3x-4)}{(x-1)^2(x-2)^2}$	$\frac{(-)(-)}{(+)}$ $=(+)$ ⑤	$\frac{(+)(-)}{(+)}$ $=(-)$ ⑤	$\frac{(+)(-)}{(+)}$ $=(-)$ ⑤	$\frac{(+)(+)}{(+)}$ $=(+)$ ⑤	$\frac{(+)(+)}{(+)}$ $=(+)$ ⑤
$f(x)$	↗	∩ ↘	↘ ∪	↗	↗

⑩

Increasing on : $(-\infty,\ 0]$, $[\frac{4}{3},\ 2)$ and $(2,\ \infty)$

Decreasing on : $(0,\ 1]$ and $[1,\ \frac{4}{3})$

$\boxed{50}$

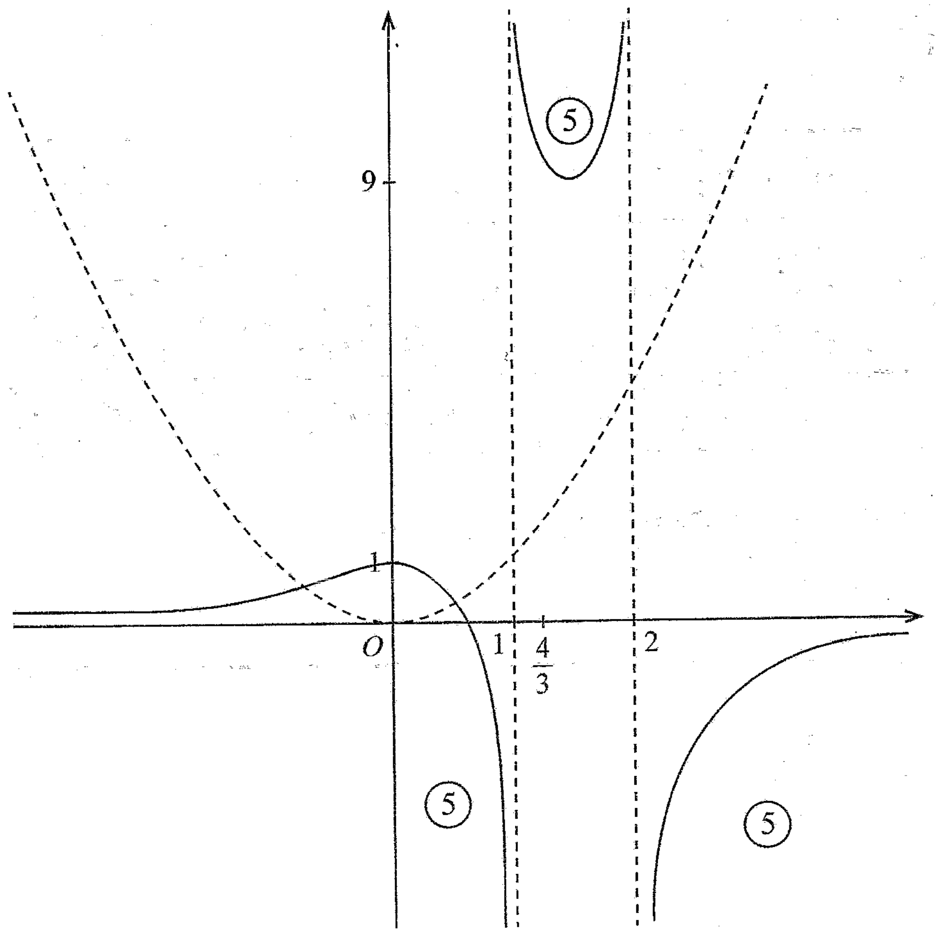
Local maximum at $(0,\ 1)$ ⑤

Local minimum at $(\frac{4}{3},\ 9)$ ⑤

$\lim_{x\to 1-} f(x) = -\infty$, $\lim_{x\to 1+} f(x) = \infty$ ⑤

$\lim_{x\to 2-} f(x) = \infty$, $\lim_{x\to 2+} f(x) = -\infty$

$\lim_{x\to \pm\infty} f(x) = 0$. ⑤



$\boxed{35}$

$x^2(x-1)(x-2) = 2 - 3x$

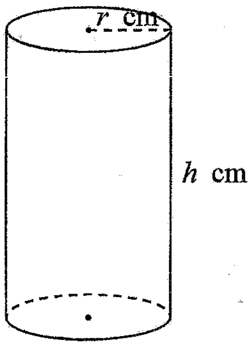
$\Leftrightarrow x^2 = \dfrac{2-3x}{(x-1)(x-2)}$ ⑤

$\Leftrightarrow x^2 = f(x)$

$\therefore$ The number of real solutions = 2. ⑤

$\boxed{10}$

(*b*)



$S=2\pi rh \quad 2\pi r^{2}+$

$\pi r^{2}h=1024\pi$

$\therefore \quad h=\dfrac{1024}{r^{2}}$ ⑤

$\therefore \quad S=2\pi r\,\dfrac{1024}{r^{2}}+\;2\pi r^{2}$

$=2\pi\left(\dfrac{1024}{r}\quad r^{2}\right)$ ⑤

$\boxed{10}$

$\dfrac{dS}{dr}=2\pi\left(-\dfrac{1024}{r^{2}}+\;2r\right)$ ⑤

$\dfrac{dS}{dr}=0$ ⑤ $\Rightarrow \dfrac{1024}{r^{2}}=2r$

$\Leftrightarrow r^{3}=512$

$\Leftrightarrow r=8.$ ⑤

$\dfrac{dS}{dr}<0$ for $0<r<8.$

$\dfrac{dS}{dr}>0$ for $r>8.$

$\therefore$ S is minimum when $r=8.$ ⑤

$\boxed{20}$

15. (*a*) Find the values of the real constants A and B such that $3t^2+4 = A(t^2-2t+4)+Bt(t+1)$ for all $t \in \mathbb{R}$.

Hence or otherwise, find $\displaystyle\int \frac{3t^2+4}{(t+1)(t^2-2t+4)}\,dt$.

(*b*) Using the substitution $u = x+\sqrt{x^2+3}$, show that $\displaystyle\int_0^1 \frac{1}{\sqrt{x^2+3}}dx = \frac{1}{2}\ln 3$.

Let $J = \displaystyle\int_0^1 \sqrt{x^2+3}\,dx$. Using integration by parts, show that $2J = 2 + \displaystyle\int_0^1 \frac{3}{\sqrt{x^2+3}}dx$.

Deduce that $J = 1+\frac{3}{4}\ln 3$.

(*c*) Using the formula $\displaystyle\int_0^a f(x)dx = \int_0^a f(a-x)dx$, where a is a constant, show that

$$\int_0^{\frac{\pi}{4}} \ln\left(\frac{\cos x}{\cos x+\sin x}\right)dx = \frac{\pi}{8}\ln\left(\frac{1}{2}\right).$$

(*a*) $3t^2+4 = A(t^2-2t+4)+Bt(t+1)$

$= (A+B)t^2+(-2A+B)t+4A$ ⑤

Coefficients of t^2 : $3 = A+B$

Coefficients of t^1 : $0 = -2A+B$

Coefficients of t^0 : $4 = 4A$

$\therefore\ A = 1$ and $B = 2$.

⑤ ⑤

$\boxed{15}$

$$\therefore \quad \frac{3t^2+4}{(t+1)(t^2-2t+4)} = \frac{1}{t+1} \quad \frac{2t}{t^2-2t+4}. \quad ⑩$$

$$\int \frac{3t^2+4}{(t+1)(t^2-2t+4)}\,\mathrm{d}t = \int \left\{ \frac{1}{t+1} \quad \frac{2t-2+2}{t^2-2t+4} \right\} \mathrm{d}t$$

$$= \int \frac{1}{t+1}\,\mathrm{d}t \quad \int \frac{2t-2}{t^2-2t+4}\,\mathrm{d}t \quad 2\int \frac{1}{(t-1)^2+3}\,\mathrm{d}t$$

$$= \ln|t \quad 1| \quad \ln|t^2 \quad 2t \quad 4| \quad \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{t-1}{\sqrt{3}}\right) + C, \quad ⑤$$

⑤ ⑤ ⑤

where C is an arbitrary constant.

$\boxed{30}$

(*b*) $u = x + \sqrt{x^2+3}$.

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 1 \quad \frac{x}{\sqrt{x^2+3}} \quad \frac{u}{\sqrt{x^2+3}} \quad ⑤$$

$$\therefore \quad \frac{1}{\sqrt{x^2+3}}\,\mathrm{d}x = \frac{1}{u}\,\mathrm{d}u$$

$x = 0 \Rightarrow u = \sqrt{3}$. ⑤

$x = 1 \Rightarrow u = 3$.

$$\text{So,} \quad \int_0^1 \frac{1}{\sqrt{x^2+3}}\,\mathrm{d}x = \int_{\sqrt{3}}^3 \frac{1}{u}\,\mathrm{d}u = \ln|u|\Big|_{\sqrt{3}}^3 = \ln 3 \quad \ln\sqrt{3}$$

⑤ ⑤

$$= \ln\sqrt{3} \quad \frac{1}{2}\ln 3. \quad ⑤$$

$\boxed{25}$

$$J = \int_0^1 \sqrt{x^2+3}\,\mathrm{d}x = x\sqrt{x^2+3}\Big|_0^1 - \int_0^1 \frac{x^2}{\sqrt{x^2+3}}\,\mathrm{d}x \quad ⑩$$

$$= 2 \quad \int_0^1 \frac{x^2+3-3}{\sqrt{x^2+3}}\,\mathrm{d}x \quad ⑤$$

$$- \therefore \quad J = 2 \int_0^1 +\sqrt{x^2+3}\,\mathrm{d}x \quad \int_0^1 \frac{3}{\sqrt{x^2+3}}\,\mathrm{d}x \quad ⑩$$

$$\Rightarrow 2J = \cancel{2} \int_0^1 \frac{3}{\sqrt{x^2+3}}\,\mathrm{d}x . \quad ⑤$$

$$\boxed{30}$$

$$\therefore \quad J = 1+ \frac{3}{2}\ \frac{1}{2}\ln 2$$

$$= 1 \quad \frac{3}{4}\ln 2 \quad ⑩$$

$$\boxed{10}$$

$$(c) \text{ Let } I = \int_0^{\frac{\pi}{4}} \ln\left(\frac{\cos x}{\cos x + \sin x}\right)\mathrm{d}x .$$

$$\text{Then } I = \int_0^{\frac{\pi}{4}} \ln\left(\frac{\cos\left(\frac{\pi}{4}-x\right)}{\cos\left(\frac{\pi}{4}-x\right)+\sin\left(\frac{\pi}{4}-x\right)}\right)\mathrm{d}x \quad ⑤$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{\cos\frac{\pi}{4}\cos x + \sin\frac{\pi}{4}\sin x}{\cos\frac{\pi}{4}\cos x + \sin\frac{\pi}{4}\sin x + \sin\frac{\pi}{4}\cos x - \cos\frac{\pi}{4}\sin x}\right)\mathrm{d}x \quad ⑩$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{\cos x + \sin x}{2\cos x}\right)\mathrm{d}x \quad ⑤$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{1}{2}\right)\mathrm{d}x \quad \int_0^{\frac{\pi}{4}} \ln\left(\frac{\cos x}{\cos x + \sin x}\right)\mathrm{d}x \quad ⑩$$

$$\therefore \quad = I \quad \ln\left(\frac{1}{2}\right)x\Big|_0^{\frac{\pi}{4}} \quad I. - \quad ⑤$$

$$\Rightarrow 2I = \ln\left(\frac{1}{2}\right)\ \frac{\pi}{4}$$

$$\therefore \quad I = \frac{\pi}{8}\ln\left(\frac{1}{2}\right). \quad ⑤$$

$$\boxed{40}$$

16. Let $A \equiv (1, 2)$ and $B \equiv (a, b)$, where $a, b \in \mathbb{R}$. It is given that the perpendicular bisector l of the line segment AB has the equation $x + y - 4 = 0$. Find the values of a and b.

Let $C \equiv (3, 1)$. Show that the point C lies on the line l and find $A\hat{C}B$.

Let S be the circle through the points A, B and C. Show that the centre of S is given by $\left(\frac{13}{6}, \frac{11}{6}\right)$ and find the equation of S.

Hence, find the equation of the circle passing through the points A, B and the point $D \equiv (0, 3)$.

The mid-point of $AB \equiv \left(\frac{1+a}{2}, \frac{2+b}{2}\right)$. ⑤

This point lies on l.

$\therefore \frac{1+a}{2} + \frac{2+b}{2} - 4 = 0$

$\therefore a + b = 5$ ——— ① ⑤

Also, l is perpendicular to AB.

⑤

$\therefore \left(\frac{b-2}{a-1}\right) \times (-1) = -1.$ ⑤

$\therefore b - 2 = a - 1$

$\therefore b - a = 1$ ——— ② ⑤

Now, ① and ② give us $a = 2$ and $b = 3$.
⑤ ⑤

So, $A \equiv (1, 2)$, $B \equiv (2, 3)$ and $C \equiv (3, 1)$.

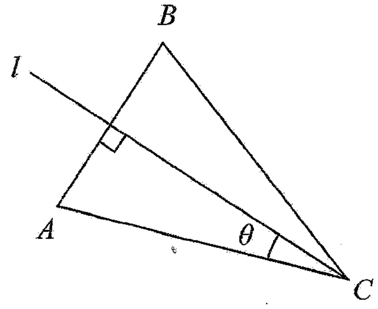
$\boxed{35}$

Substitute $x = 3$ and $y = 1$ in $x + y - 4 = 0$. ⑤

L.H.S. $= 3 + 4 - 1$

$= 0 =$ R.H.S.

$\therefore C$ lies on l. ⑤



Let the angle between AC and l be θ.

Then θ is acute.

The slope of $AC = -\frac{1}{2}$. ⑤

$$\tan\theta = \left| \frac{-\frac{1}{2}-(-1)}{1+\left(-\frac{1}{2}\right)\times(-1)} \right| = \frac{1}{3} \quad ⑩$$

$\theta = \tan^{-1}\left(\frac{1}{3}\right)$. ⑤

$\therefore \ A\hat{C}B = 2\theta \quad 2\tan^{-1}\left(\frac{1}{3}\right)$. ⑤

$\boxed{35}$

Let m be the perpendicular bisector of AC.

The slope of $m = 2$. ⑤

The mid-point of $AC \equiv \left(2, \frac{3}{2}\right)$. ⑤

The equation of m: $y-\frac{3}{2}=2(x \quad 2)$ ⑤

i.e. $y-2x+\frac{5}{2}=0$.

The centre of S is the point of intersection of l and m. ⑤

Solving $x+y-4=0$ and $y-2x+\frac{5}{2}=0$ simultaneously, ⑤ we get

$3x = 4 \quad \frac{5}{2}$.

$\therefore \ x=\frac{16}{3}$ and $y = 4 \quad \frac{13}{6} = \frac{11}{6}$.

⑤ ⑤

$\therefore$ The centre of $S \equiv \left(\frac{13}{6}, \frac{11}{6}\right)$.

The radius of $S = \sqrt{\left(\frac{13}{6}-1\right)^2+\left(\frac{11}{6}-2\right)^2}$ ⑩

$= \sqrt{\frac{49}{36}+\frac{1}{36}}$

$= \sqrt{\frac{25}{18}}$ ⑤

$\therefore$ The equation of S: $\left(x-\frac{13}{6}\right)^2+\left(y-\frac{11}{6}\right)^2=\frac{25}{18}$. ⑩

$x^2+y^2-\frac{13}{3}x-\frac{11}{3}y+\frac{20}{3}=0$

$\boxed{60}$

$\therefore$ The equation of AB: $y-2=1(x-1)$ ⑤

i.e. $x-y+1=0$

The equation of any circle through A and B is given by $\left(x-\frac{13}{6}\right)^2+\left(y-\frac{11}{6}\right)^2-\frac{25}{18}+\lambda(x-y+1)=0$, where $\lambda \in \mathbb{R}$. ⑩

For this to go through $D \equiv (0, 3)$, we must have $\left(\frac{13}{6}\right)^2+\left(\frac{7}{6}\right)^2-\frac{25}{18}+\lambda(-2)=0$.

$\lambda=\frac{21}{9} \quad \frac{7}{3}=$ ⑤

$\therefore$ The required equation is $x^2+y^2-2x-6y+9=0$.

$\boxed{20}$

---

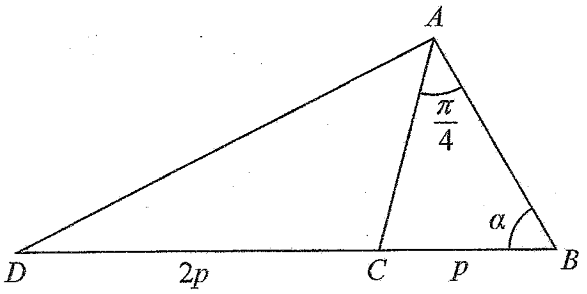
17. (*a*) Express $6\cos 2x - 8\sin 2x$ in the form $R\cos(2x+\alpha)$, where $R>0$ and $0<\alpha<\frac{\pi}{2}$.

Hence, solve $6\cos 2x - 8\sin 2x = 5$.

Express $24\cos^2 x - 32\sin x\cos x$ in the form $a\cos 2x + b\sin 2x + c$, where a, b, c $(\in \mathbb{R})$ are constants to be determined.

Deduce the minimum value of $24\cos^2 x - 32\sin x\cos x$.

(*b*)



In the triangle ABC shown in the figure, $BC = p$, $B\hat{A}C = \frac{\pi}{4}$ and $A\hat{B}C = \alpha$. The point D lies on the extended line BC such that $CD = 2p$.

Show that $AB = p(\cos\alpha + \sin\alpha)$.

Find AD^2 in terms of p and α.

Deduce that if $AD = 3p$, then $\alpha = \tan^{-1}(5)$.

(*c*) Solve the equation $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$.

(*a*) $6\cos 2x - 8\sin 2x$ $\sqrt{6^2+8^2} = 10$

$$= 10\left(\frac{6}{10}\cos 2x - \frac{8}{10}\sin 2x\right) \quad ⑤$$

$$= 10\left(\frac{3}{5}\cos 2x - \frac{4}{5}\sin 2x\right)$$

$= 10(\cos\alpha\cos 2x - \sin\alpha\sin 2x)$, where $0<\alpha<\frac{\pi}{2}$ is such that $\cos\alpha = \frac{3}{5}$ and $\sin\alpha = \frac{4}{5}$. ⑤
⑤

$= 10\cos(2x \quad \alpha)$. ⑤

$\therefore R = 10$ and $\alpha = \tan^{-1}\left(\frac{4}{3}\right)$. ⑤

$\boxed{25}$

$6\cos 2x - 8\sin 2x = 5$

$10\cos(2x+\alpha) = 5$ ⑤

$\therefore\ \cos(2x+\alpha) = \frac{1}{2} = \cos\frac{\pi}{3}$ ⑤

$2x+\alpha = 2n\pi \pm \frac{\pi}{3}$, where $n \in \mathbb{Z}$. ⑤

$\therefore\ x = n\pi\ \frac{\alpha}{2} \pm \frac{\pi}{6}$, where $n \in \mathbb{Z}$. ⑤

$\boxed{20}$

$24\cos^2 x - 32\sin x\cos x$

$= 24\left(\frac{\cos 2x + 1}{2}\right) - 32 \times \frac{1}{2}\sin 2x$ ⑩

$= 12\cos 2x - 16\sin 2x + 12$

$\therefore\ a = 12,\quad b = -16$ and $c = 12.$ ⑤

$\boxed{15}$

$24\cos^2 x - 32\sin x\cos x$

$= 12\cos 2x - 16\sin 2x + 12$

$= 2(6\cos 2x - 8\sin 2x) + 12$

$= 20\underbrace{\cos(2x+\alpha)}_{\min\ =\ -1} + 12$ ⑤

$\therefore$ The required minimum $= -20 + 12$

$= -8.$ ⑤

$\boxed{10}$

(*b*) Sine Rule for the $\triangle ABC$:

$$\frac{AB}{\sin\left(\frac{\pi}{4}+\alpha\right)}=\frac{p}{\sin\frac{\pi}{4}} \quad ⑩$$

$$\therefore \quad AB=\sqrt{2}p\left(\sin\frac{\pi}{4}\cos\alpha+\cos\frac{\pi}{4}\sin\alpha\right) \quad ⑤$$

$$=\sqrt{2}p\left(\frac{1}{\sqrt{2}}\cos\alpha+\frac{1}{\sqrt{2}}\sin\alpha\right)$$

$$=p(\cos\alpha \quad \sin\alpha). \quad ⑤$$

$\boxed{20}$

Cosine Rule for the $\triangle ABD$:

$$AD^2=AB^2+BD^2-2AB\cdot BD\cos\alpha \quad ⑤$$

$$=p^2(\cos\alpha \quad \sin\alpha)^2 \quad 9p^2+ \quad 2p(\cos\alpha \quad \sin\alpha)\ 3p\cos\alpha \quad ⑤$$

Suppose that $AD = 3p$.

Then $9p^2=p^2(\cos\alpha \quad \sin\alpha)^2 \quad 9p^2+ \quad 6p^2(\cos\alpha \quad \sin\alpha)\ \cos\alpha$ ⑤

$$\therefore \quad (\cos\alpha+\sin\alpha)^2-6(\cos\alpha+\sin\alpha)\cos\alpha=0$$

$$\Rightarrow \quad \cos\alpha+\sin\alpha=6\cos\alpha \quad ⑤ \qquad (0<\alpha<\frac{\pi}{2})$$

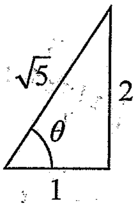
$$\Rightarrow \quad \sin\alpha=5\cos\alpha$$

$$\Rightarrow \quad \tan\alpha=5$$

$$\Rightarrow \quad \alpha=\tan^{-1}(5). \quad ⑤$$

$\boxed{25}$

(*c*) $\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) = \tan^{-1}(2).$ ⑤



Let $\alpha = \tan^{-1}(x+1)$ and $\beta = \tan^{-1}(x-1).$

$\alpha + \beta = \tan^{-1}(2)$ ⑤

$\Rightarrow \tan(\alpha + \beta) = 2$ ⑤

$\Rightarrow \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = 2$ ⑤

$\Rightarrow \frac{x+1+x-1}{1-(x^2-1)} = 2$ ⑤ $\Rightarrow x = 2 - x^2$

$\Rightarrow x^2 + x - 2 = 0$

$\Rightarrow (x-1)\ (x+2) = 0$

$x = 1$ or $x = -2.$ ⑤

$x = -2$ is not a solution. $x = 1$ is a solution.

$\therefore\ x = 1.$ ⑤

$\boxed{35}$

Confidential

Department of Examinations - Sri Lanka

G.C.E. (A/L) Examination - 2024

10 – Combined Mathematics II

Marking Scheme

This has been prepared for the use of marking examiners. Changes would be made according to the views presented at the Chief/Assistant Examiners' meeting.

Amendments to be included.

G. C. E (Advanced Level) Examination – 2024

10 - Combined Mathematics II

Distribution of Marks

Paper I

Part A	=	**10 × 25**	=	**250**
Part B	=	**05 × 150**	=	**750**

Total $= \dfrac{\mathbf{1000}}{\mathbf{10}}$

Final marks = **100**

Common Techniques of Marking Answer Scripts.

It is compulsory to adhere to the following standard method in marking answer scripts and entering marks into the mark sheets.

1. Use a red color ball point pen for marking. (Only Chief/Additional Chief Examiner may use a mauve color pen.)
2. Note down Examiner's Code Number and initials on the front page of each answer script.
3. Write off any numerals written wrong with a clear single line and authenticate the alterations with Examiner's initials.
4. Write down marks of each subsection in a △ and write the final marks of each question as a rational number in a □ with the question number. Use the column assigned for Examiners to write down marks.

Example: **Question No. 03**

(i) .... √ $\frac{4}{5}$
....
....

(ii) ....
.... √ $\frac{3}{5}$
....

(iii) ....
.... √ $\frac{3}{5}$
....

03 (i) $\frac{4}{5}$ + (ii) $\frac{3}{5}$ + (iii) $\frac{3}{5}$ = $\boxed{\frac{10}{15}}$

MCQ answer scripts: (Template)

1. Marking templets for G.C.E.(A/L) and GIT examination will be provided by the Department of Examinations itself. Marking examiners bear the responsibility of using correctly prepared and certified templates.
2. Then, check the answer scripts carefully. If there are more than one or no answers Marked to a certain question write off the options with a line. Sometimes candidates may have erased an option marked previously and selected another option. In such occasions, if the erasure is not clear write off those options too.
3. Place the template on the answer script correctly. Mark the right answers with a 'V' and the wrong answers with a 'X' against the options column. Write down the number of correct answers inside the cage given under each column. Then, add those numbers and write the number of correct answers in the relevant cage.

Structured essay type and assay type answer scripts:

1. Cross off any pages left blank by candidates. Underline wrong or unsuitable answers. Show areas where marks can be offered with check marks.

2. Use the right margin of the overland paper to write down the marks.
3. Write down the marks given for each question against the question number in the relevant cage on the front page in two digits. Selection of questions should be in accordance with the instructions given in the question paper. Mark all answers and transfer the marks to the front page, and write off answers with lower marks if extra questions have been answered against instructions.

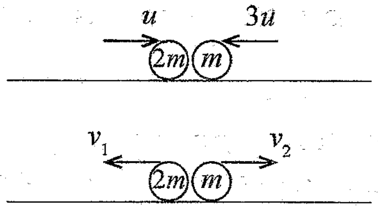
4. Add the total carefully and write in the relevant cage on the front page. Turn pages of answer script and add all the marks given for all answers again. Check whether that total tallies with the total marks written on the front page.

Preparation of Mark Sheets.

Except for the subjects with a single question paper, final marks of two papers will not be calculated within the evaluation board this time. Therefore, add separate mark sheets for each of the question paper. Write paper 01 marks in the paper 01 column of the mark sheet and write them in words too. Write paper II Marks in the paper II Column and wright the relevant details.

***

1. A particle A of mass $2m$ and a particle B of mass m moving on a smooth horizontal table along the same straight line towards each other with speeds u and $3u$ respectively, collide directly. After the collision A and B move in opposite directions. The coefficient of restitution between A and B is e. Show that $e > \frac{1}{8}$.



For the system $\mathbf{I} = \Delta(m\mathbf{v})$: $\longrightarrow$

$$0 = -2mv_1 + mv_2 - (2mu - 3mu) \quad ⑤$$

$$2v_1 - v_2 = u \quad \text{-----------} (1)$$

Newton's Law of Restitution:

$$v_2 + v_1 = e(u + 3u) \quad ⑤$$

$$v_1 + v_2 = 4eu \quad \text{-----------} (2)$$

(1) and (2) $\Rightarrow 3v_1 = u + 4eu$

$$\therefore \; v_1 = \frac{1}{3}(1 + 4e)u > 0 \quad ⑤$$

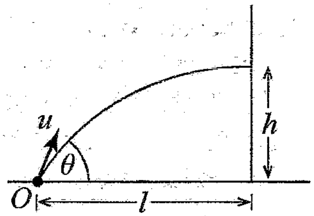
$$-3v_2 = u - 8eu$$

$$\therefore \; v_2 = \frac{1}{3}(8e - 1)u \quad ⑤$$

For $v_2 > 0$, we must have $e > \frac{1}{8}$. ⑤

2. A particle is projected from a point O on a horizontal ground, with an initial speed u at an angle $\theta\left(0<\theta<\frac{\pi}{2}\right)$ to the horizontal. The particle hits a vertical wall which is at a horizontal distance l from O at a height $h\ (>0)$ from the ground (see the figure).

Show that $h=l\tan\theta \quad \frac{gl^2}{2u^2}\sec^2\theta$ and **deduce** that $\sin 2\theta>\frac{gl}{u^2}$.



$$S=ut \quad \frac{1}{2}at^2 : \;\rightarrow\; l=u\cos\theta\, t$$ ⑤

$$\uparrow\; h=u\sin\theta\; t \quad \frac{1}{2}gt^2$$ ⑤

$$\therefore\; h=u\sin\theta\cdot\frac{l}{u\cos\theta} \quad \frac{1}{2}g\frac{l^2}{u^2\cos^2\theta}$$

$$\Rightarrow\; h=l\tan\theta \quad \frac{gl^2}{2u^2}\sec^2\theta$$ ⑤

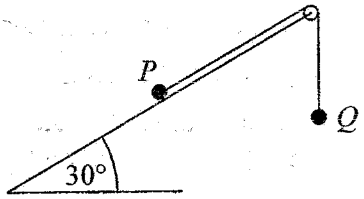
$$h>0 \;\Rightarrow\; l\tan\theta>\frac{gl^2}{2u^2}\sec^2\theta$$ ⑤

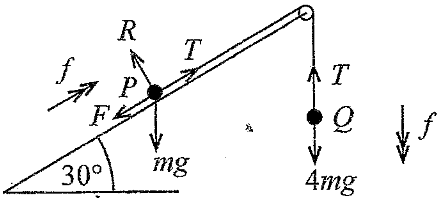
$$\Rightarrow\; \frac{2\tan\theta}{1+\tan^2\theta}>\frac{gl}{u^2}$$

$$\Rightarrow\; \frac{2\sin\theta\cos\theta}{\cos^2\theta+\sin^2\theta}>\frac{gl}{u^2}$$

$$\Rightarrow\; \sin 2\theta>\frac{gl}{u^2}.$$ ⑤

3. A particle P of mass m lies on a rough inclined plane whose inclination to the horizontal is 30°. Particle P is connected, by a light inextensible string passing over a fixed smooth pulley at the top of the inclined plane, to a particle Q of mass $4m$ which is free to move vertically (see the figure). The part of the string on the inclined plane lies along a line of greatest slope of the plane. The coefficient of friction between P and the plane is $\frac{1}{2}$. The system is released from rest with the string taut. It is given that P moves up the inclined plane. Obtain equations sufficient to determine the tension of the string.





⑤

$F = ma$:

Ⓟ ↖ $R - mg\cos 30^\circ = 0$ ⑤

$\therefore \; R = \frac{\sqrt{3}}{2}mg$

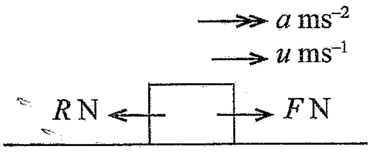
$F = \frac{1}{2}R \quad \frac{\sqrt{3}}{4}mg$ ⑤

Ⓟ ↗ $T - F - mg\cos 60^\circ = mf$ ⑤

$\therefore \; T - F - \frac{mg}{2} = mf$

Ⓠ ↓ $4mg - T = 4mf$ ⑤

4. A car of mass M kg travels along a horizontal straight road with its engine working at a constant power of P W. There is a constant resistance of R N to the motion of the car. Find the acceleration of the car at the instant when its speed is u m s^{-1}.
Now, the car travels at a constant speed up a straight road that is inclined at an angle $\theta \left(0 < \theta < \frac{\pi}{2}\right)$ to the horizontal. Find this constant speed if the car is subjected to the same resistance R N and have the same power P W.



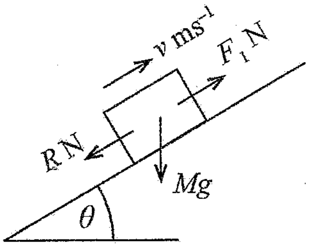
$$P = F \cdot u$$

$$\therefore \; F = \frac{P}{u} \quad ⑤$$

$F = ma$: →

$$F - R = M \cdot a \quad ⑤$$

$$\therefore \; a = \frac{1}{M}\left(\frac{P}{u} \quad R\right) \text{ms}^{-2}. \quad ⑤$$



$$P = F_1 \quad v:$$

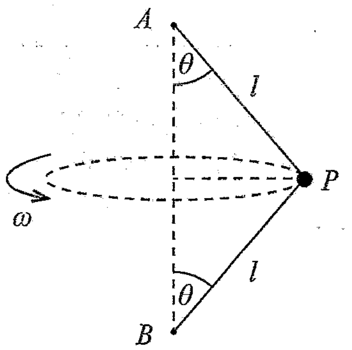
$$\therefore \; F_1 = \frac{P}{v}$$

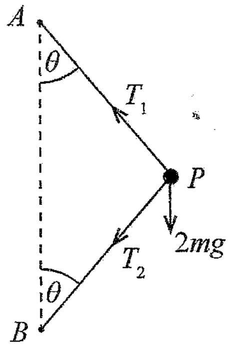
$F = ma$: ↗ $F_1 - R - Mg\sin\theta = 0$ ⑤

$$\frac{P}{v} = R \quad Mg\sin\theta$$

$$\therefore \; v = \frac{P}{R + Mg\sin\theta} \text{ ms}^{-1}. \quad ⑤$$

5. A particle P of mass $2m$ is connected to two fixed points A and B lying on a vertical line by two light inextensible strings each of length l. The particle P moves in a horizontal circle with a constant angular velocity ω with both strings taut and making an angle θ $\left(0<\theta<\frac{\pi}{2}\right)$ to the vertical, as shown in the figure.
Show that the tension in the string AP is $m(l\omega^2+g\sec\theta)$.





$F=ma$:

Ⓟ ↑ $T_1\cos\theta - T_2\cos\theta - 2mg = 0$ ⑤

$\therefore\ T_1 - T_2 = 2mg\sec\theta$ ——————— (1) ⑤

Ⓟ ← $T_1\sin\theta + T_2\sin\theta = 2ml\sin\theta\,\omega^2$ ⑤

$\therefore\ T_1 + T_2 = 2ml\omega^2$ ——————— (2) ⑤

(1) and (2) ⇒ $2T_1 = 2mg\sec\theta \quad 2ml\omega^2 \quad +$

$\therefore\ T_1 = m(l\omega^2 \quad g\sec\theta)$. ⑤

6. Let $\mathbf{u}$ and $\mathbf{v}$ be two unit vectors such that $\mathbf{u}\cdot\mathbf{v}=\frac{1}{2}$. Also, let $\mathbf{a}=\alpha\mathbf{u}+\mathbf{v}$ and $\mathbf{b}=\mathbf{u}+\beta\mathbf{v}$, where $\alpha, \beta \in \mathbb{R}$. If the vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, and $\mathbf{a}+\mathbf{b}$ is parallel to $\mathbf{u}$, find the values of α and β.

$$\mathbf{a}\cdot\mathbf{b} = (\alpha\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\beta\mathbf{v})$$

$$0 = \alpha\,\mathbf{u}\cdot\mathbf{u}+\alpha\beta\,\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{u}+\beta\,\mathbf{v}\cdot\mathbf{v}$$

$$0 = \alpha+\alpha\beta\cdot\frac{1}{2}+\frac{1}{2}+\beta \text{ ---------- (1)} \quad ⑤$$

$$\mathbf{a}+\mathbf{b} = \lambda\mathbf{u} \quad , \quad \text{where } \lambda \in \mathbb{R}. \text{ ---------- (2)}$$

$$(2)\cdot\mathbf{u} \Rightarrow (\alpha\mathbf{u}+\mathbf{v})\cdot\mathbf{u}+(\mathbf{u}+\beta\mathbf{v})\cdot\mathbf{u}=\lambda\mathbf{u}\cdot\mathbf{u}$$

$$\alpha+\frac{1}{2}+1+\beta\cdot\frac{1}{2}=\lambda \text{ ---------- (3)} \quad ⑤$$

$$(2)\cdot\mathbf{v} \Rightarrow (\alpha\mathbf{u}+\mathbf{v})\cdot\mathbf{v}+(\mathbf{u}+\beta\mathbf{v})\cdot\mathbf{v}=\lambda\mathbf{u}\cdot\mathbf{v}$$

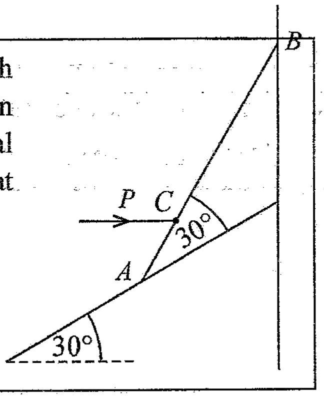
$$\frac{\alpha}{2}+1+\frac{1}{2}+\beta=\frac{\lambda}{2} \text{ ---------- (4)} \quad ⑤$$

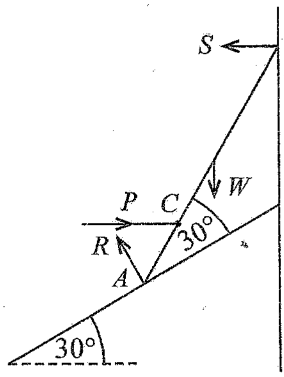
$$(3) \text{ and } (4) \Rightarrow \quad \alpha+\frac{3}{2}+\frac{\beta}{2}=\alpha+3+2\beta$$

$$\Rightarrow \quad \beta=-1. \quad ⑤$$

Now, (1) $\Rightarrow$ $\alpha=1.$ ⑤

7. A uniform rod AB of length $4a$ and weight W is kept in equilibrium with its upper end B against a smooth vertical wall and the lower end A on a smooth plane inclined at 30° to the horizontal by applying a horizontal force P to the rod at the point C, where $AC = a$. The rod is inclined at 30° to the inclined plane, as shown in the figure. Find the value of P.





⑤

$\longrightarrow \quad P - S - R\sin 30^\circ = 0$ ⑤

$\therefore \quad P = S + \frac{R}{2}$

$\uparrow \quad R\cos 30^\circ - W = 0$ ⑤

$\therefore \quad R = \frac{2W}{\sqrt{3}}$

$\overset{\curvearrowright}{A} \quad P \times a\sin 60^\circ + W \times 2a\cos 60^\circ - S \times 4a\sin 60^\circ = 0$ ⑤

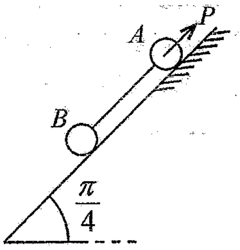
$$P \times \frac{\sqrt{3}}{2} + W - \left(P - \frac{R}{2}\right)2\sqrt{3} = 0$$

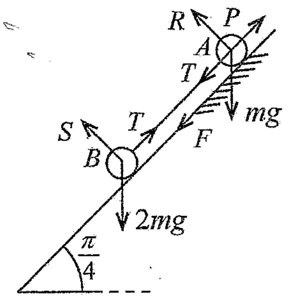
$$\sqrt{3}P + 2W - 4\sqrt{3}P + \frac{2W}{\sqrt{3}} \times 2\sqrt{3} = 0$$

$$3\sqrt{3}P = 6W.$$

$P = \frac{2W}{\sqrt{3}}.$ ⑤

8. Two particles A and B of masses m and $2m$ respectively, are placed on a plane inclined at an angle $\frac{\pi}{4}$ to the horizontal and are connected by a light inextensible string and kept in equilibrium by a force P applied to A, as shown in the figure. The line of action of P and the string lie along a line of greatest slope of the plane. The particle A lies on the rough part of the plane and the particle B lies on the smooth part of the plane. The coefficient of friction between A and the plane is $\frac{1}{2}$.
Show that $2\left|\sqrt{2}\,P-3mg\right| \leq mg$.





⑤

For the system: ↗ $P-F-mg\sin\frac{\pi}{4}-2mg\sin\frac{\pi}{4}=0.$ ⑤

$$\therefore \quad F=-P \quad 3mg \ \frac{1}{\sqrt{2}}$$

Ⓐ ↖ $R-mg\cos\frac{\pi}{4}=0$ ⑤

$$\therefore \quad R=\frac{mg}{\sqrt{2}}$$

For the equilibrium:

$$\frac{1}{2} \geq \frac{|F|}{R}$$ ⑤

$$\therefore \quad \left|P-\frac{3mg}{\sqrt{2}}\right| \leq \frac{mg}{2\sqrt{2}}$$ ⑤

$$\therefore \quad 2\left|\sqrt{2}P-3mg\right| \leq mg .$$

9. Let A and B be two events of a sample space Ω. It is given that $P(A) = \frac{1}{5}$, $P(A \mid B) = \frac{1}{10}$ and $P(B \mid A) = \frac{3}{10}$. Find $P(B)$ and $P(A \cup B)$.

$$P(B \mid A) = \frac{3}{10}$$

$$\therefore \frac{P(B \cap A)}{P(A)} = \frac{3}{10}$$ ⑤

$$\therefore P(B \cap A) = \frac{3}{10} \times \frac{1}{5} = \frac{3}{50}$$ ⑤

$$P(A \mid B) = \frac{1}{10}$$

$$\therefore \frac{P(A \cap B)}{P(B)} = \frac{1}{10}$$ ⑤

$$\therefore P(B) = 10 \times P(A \cap B) = 10 \times \frac{3}{50} = \frac{3}{5}$$ ⑤

$$P(A \cup B) = P(A) \quad P(B) \quad P(A \quad B)$$

$$= \frac{1}{5} + \frac{3}{5} - \frac{3}{50} = \frac{37}{50}.$$ ⑤

10. The median, the mode and the mean of the following seven observations, arranged in the ascending order, are 5, 7 and 5 respectively:

$$1,\ 3,\ 4,\ p,\ q,\ r,\ s$$

Here p, q, r and s are real numbers.

Find the values of p, q, r and s, and show that the variance of the seven observations is $\frac{38}{7}$.

Median = 5.

$\therefore\ p = 5.$ ⑤

Mode = 7.

$\therefore$ At least two of q, r, s must be $7^{\text{'s}}$. ⑤

Let us take 2 of them to be $7^{\text{'s}}$ and the other one be a.

Mean = 5

$\therefore\ 5 = \frac{1+3+4+5+7+7+a}{7}$ ⑤

$35 = 27 + a$

$\therefore\ a = 8.$

So, $q = r = 7$ and $s = 8.$ ⑤

$$\text{Variance} = \frac{1}{7}\left[(1-5)^2+(3-5)^2+(4-5)^2+(5-5)^2+2(7-5)^2+(8-5)^2\right]$$

$$= \frac{1}{7}[16+4+1+8+9] = \frac{38}{7}.$$ ⑤

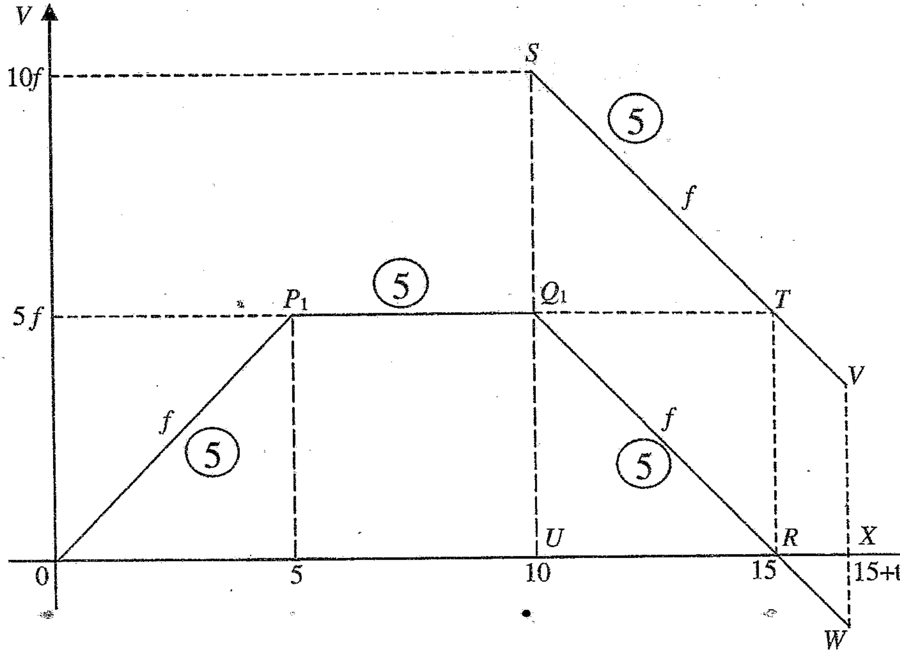
11. (*a*) A car P that begins its journey from rest on a straight road from a point O at time $t = 0$ s, moves with a constant acceleration of f m s^{-2} for 5 seconds. It then moves with the constant speed attained at $t = 5$ s for another 5 seconds and at $t = 10$ s decelerates at a constant deceleration of f m s^{-2} and comes to rest at a point A. The car P then changes its direction instantly and returns towards O with the same constant acceleration of f m s^{-2} on the same road.

Another car Q that begins its journey with an initial speed of $10f$ m s^{-1} from the point O at $t = 10$ s, moves towards car P with a constant deceleration of f m s^{-2} along the same road. It is given that the distance between P and Q when P comes to rest at A, is 125 m. Sketch velocity-time graphs for the motions of P and Q from $t = 0$ s until they meet, in the same diagram.

Show that

(i) $f = 10$,

(ii) cars P and Q meet at $t = 17.5$ s.



$\boxed{20}$

The time taken by P to reach A = 15 s ⑤

⑤ The distance travelled by Q from $t = 10$s to $t = 15$s

$$= \text{The area of } STRU = \frac{1}{2}(SU + TR) \times UR \quad ⑤$$

$$= \frac{1}{2}(10f + 5f) \times 5$$

$$= \frac{75}{2}f \text{ m} \quad ⑤$$

The distance from O to A $= \text{Area of } OP_1Q_1R$ ⑤

$$= \frac{1}{2} \times (5 + 15) \times 5f = 50f \text{ m} \quad ⑤$$

$$\text{Given}: 125 = 50f - \frac{75}{2}f \quad ⑤$$

$$250 = 25f$$

$$\therefore f = 10 \quad ⑤$$

$\boxed{40}$

Suppose that P and Q meet at $t=(15+t_0)\,s$.

They meet when Area $RTVX$ + Area RWX = 125. ⑤

i.e. when Area $RTVW$ = 125.

$$TR \times RX = 125 \quad ⑤$$

$$5f \times t_0 = 125 \quad ⑤$$

$$50 \times t_0 = 125$$

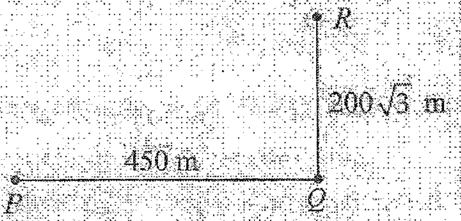
$$\therefore t_0 = \frac{125}{50} = 2.5s$$

$$\therefore P \text{ and } Q \text{ meet at } t=(15+2.5)\,s$$

$$= 17.5s. \quad ⑤$$

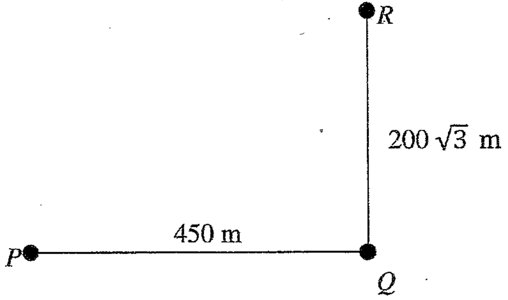
$\boxed{25}$

(*b*) Three boats P, Q and R are moving in straight-line paths with uniform speeds. At a certain instant, the boat Q is located 450 m east of the boat P and the boat R is located $200\sqrt{3}$ metres north of the boat Q (See the figure). The boat P sails with the intention of meeting the boat Q and the boat Q sails with the intention of meeting the boat R.



It is given that the boat P meets the boat Q in 45 seconds and the boat Q meets the boat R in 20 seconds.

Show that the speed of the boat P relative to the boat Q is 10 m s^{-1}, and find the distance between the boat P and the boat R at the instant when the boat Q meets the boat R.



$$\underline{\text{V}}(P,Q) = \frac{450}{45}\ \text{m s}^{-1} \rightarrow \quad ⑩$$

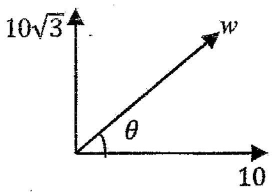
$$= 10\ \text{m s}^{-1} \rightarrow \quad ⑤$$

$\therefore$ The speed of P relative to $Q = 10$ m s^{-1}. ⑤

$$\underline{\text{V}}(Q,R) = \frac{200\sqrt{3}}{20}\text{m s}^{-1} \uparrow \quad ⑩$$

$$= 10\sqrt{3}\ \text{m s}^{-1} \uparrow \quad ⑤$$

$$\underline{V}(P,R) = \underline{V}(P,Q) + \underline{V}(Q,R)$$ ⑤



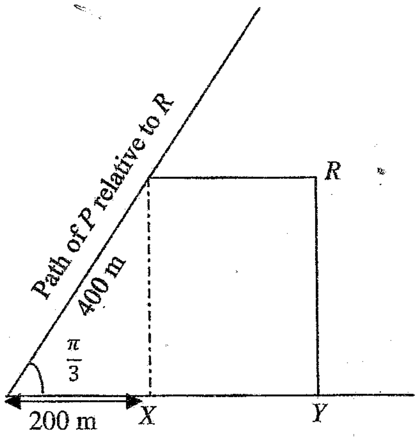
$$w = \sqrt{\left(10\sqrt{3}\right)^2 + 10^2}$$ ⑤

$$= 20 \text{ ms}^{-1}$$

$$\tan\theta = \frac{10\sqrt{3}}{10} = \sqrt{3}$$

$$\theta = \frac{\pi}{3}$$ ⑤

After 20 s :



⑩

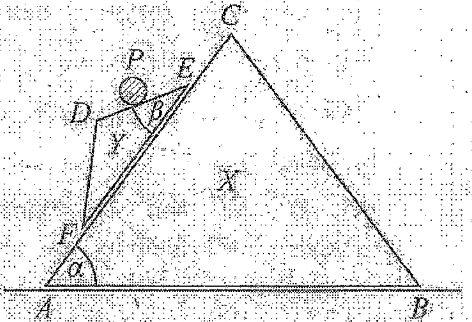
The requied distance $= XY$

$= (450 - 200)$ m

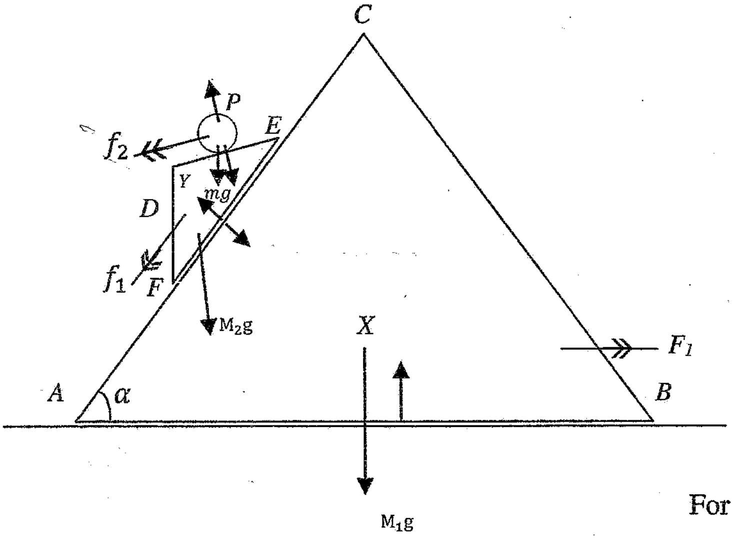
$= 250$ m. ⑤

65

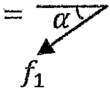
12. (*a*) The vertical cross-section through the centres of mass of two smooth uniform wedges X, Y and a particle P is shown in the figure. AC, DE and EF are lines of greatest slope of the faces containing them with $B\hat{A}C = \alpha$ and $D\hat{E}F = \beta \,(< \alpha)$. The face containing AB of the wedge X of mass M_1 is placed on a smooth horizontal table. The face containing EF of the wedge Y of mass M_2 is placed on the face of X containing AC. The particle P of mass m is placed on DE. The system is released from rest. Write down equations sufficient to determine the acceleration of the wedge X, while the wedge Y moves with its face containing EF touching the face of the wedge X containing AC and the particle P moves touching DE.



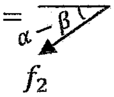
(*a*)



$\mathbf{a}(X,E) = F_1 \twoheadrightarrow$

$\mathbf{a}(Y,X) =$ 

⑩

$\mathbf{a}(P,Y) =$ 

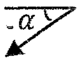
For the forces ⑩

Apply $\mathbf{F} = \mathrm{m}\mathbf{a}$:

For the system $\rightarrow$:

$$O = M_1F_1 + M_2\left(F_1 - f_1\cos\alpha\right) + m\left(F_1 - f_1\cos\alpha - f_2\cos(\alpha - \beta)\right).$$ ⑩

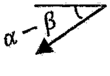
⑤ ⑤ ⑤

For Y and P 

:

$$\left(M_2 + m\right)g\sin\alpha = M_2\left(f_1 - F_1\cos\alpha\right) + m\left(f_1 + f_2\cos\beta - F_1\cos\alpha\right).$$ ⑩

⑤ ⑤ ⑤

For P 

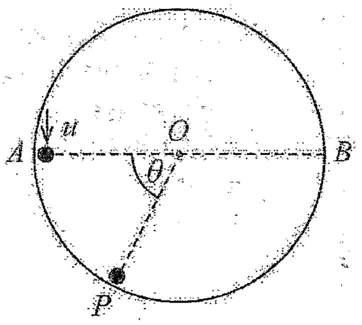
$$mg\sin(\alpha - \beta) = m\left(f_2 - f_1\cos\alpha - F_1\cos(\alpha - \beta)\right).$$ ⑩

⑤ ⑤

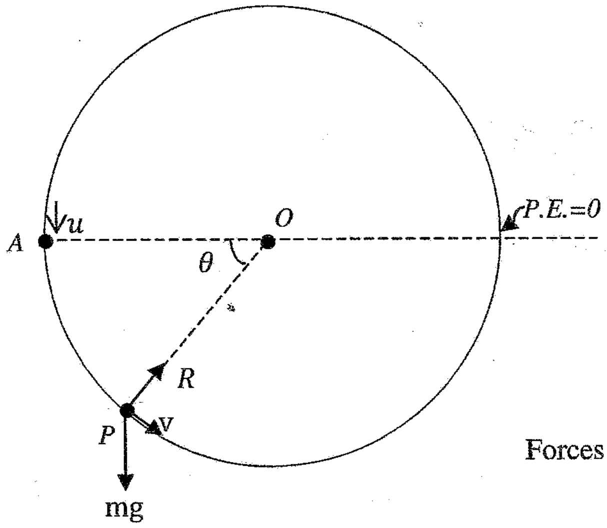
$\boxed{90}$

(*b*) The vertical cross-section perpendicular to the horizontal axis of a fixed hollow right-circular cylinder of radius a with a smooth inner surface is shown in the adjoining figure.

The point O is its centre, and A and B are the ends of its horizontal diameter. A particle P of mass m is projected in the vertically downward direction from A on the inner surface of the cylinder with speed u. Let v be the speed of P when OP has turned through an angle θ with P is in contact with the cylinder. Show that $v^2 = u^2 + 2ga\sin\theta$.



It is given that P leaves the inner surface of the cylinder when $\theta = \frac{7\pi}{6}$. Show that $u = \sqrt{\frac{3ga}{2}}$.



Forces ⑤

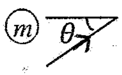
Consevation of energy :

$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 - mga\sin\theta$$

$$\therefore v^2 = u^2 + 2ga\sin\theta$$ ⑤

⑮ P.E. ⑤ K.E. ⑤ Equation ⑤

$\boxed{25}$



F = ma :

$$R - mg\sin\theta = m\frac{v^2}{a}$$ ⑩

$$\therefore R = \frac{m}{a}\left(u^2 + 2ga\sin\theta\right) + mg\sin\theta$$ ⑤

R=0 when $\theta = \frac{7\pi}{6}$. ⑤

$$\therefore \quad 0 = \frac{m}{a}\left(u^2 + 2ga\sin\frac{7\pi}{6}\right) + mg\sin\frac{7\pi}{6} \quad ⑤$$

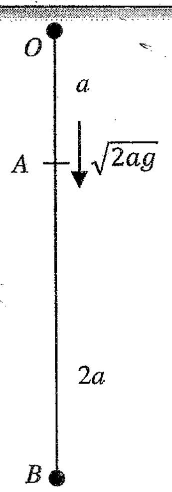
$$\therefore \quad 0 = u^2 + 2ga\left(-\frac{1}{2}\right) + ga\left(-\frac{1}{2}\right) \quad ⑤$$

$$\therefore \quad u^2 = \frac{3ga}{2}$$

$$\therefore \quad u = \sqrt{\frac{3ga}{2}}\,. \quad ⑤$$

$\boxed{35}$

13. One end of a light elastic string of natural length a is attached to a fixed point O and the other end to a particle P of mass m, and P has been set to vertical motion. When it is moving vertically downward, it passes through the point A below O, where $OA = a$, its speed is $\sqrt{2ag}$. The particle comes to instantaneous rest at point B, $3a$ below O. Show that the modulus of elasticity of the string is $\frac{3}{2}mg$.
Also, show that the equation of motion of P is given by $\ddot{x} + \omega^2\left(x - \frac{5a}{3}\right) = 0$, where $OP = x$ for $x > a$ and ω (> 0) is a constant to be determined.
Re-write the above equation of motion by taking $X = x - \frac{5a}{3}$.
Find the centre, amplitude and the period of this simple harmonic motion of the particle.
Using the formula $\dot{X}^2 = \omega^2(C^2 - X^2)$, where C is the amplitude, find the maximum speed of P.
On its way up, show that P barely reaches O.
Show that the total time taken by P to move from B to O is $\sqrt{\frac{2a}{27g}}\left(2\pi + 3\sqrt{3}\right)$.
If the above simple harmonic motion is initiated by pulling P down and releasing, state how far the string must be pulled from its natural length.



Let λ be the module of elasticity.

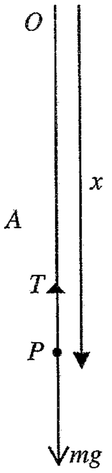
We shall apply the Principled of conservation of Energy :

$$\underset{⑤}{\frac{1}{2}m\times 2ga} + \underset{⑤}{mg\times 2a} = \underset{⑤}{\frac{1}{2}\frac{\lambda(2a)^2}{a}} \quad ⑤$$

$$3mga = 2\lambda a$$

$$\therefore \quad \lambda = \frac{3}{2}mg. \quad ⑤$$

$\boxed{25}$



$F = ma$: ↓

$$mg - T = m\ddot{x} \quad ⑤$$

$$mg - \underset{⑤}{\frac{3}{2}mg\frac{(x-a)}{a}} = m\ddot{x}$$

$$\ddot{x} = g - \frac{3}{2}g\frac{x}{a} + \frac{3g}{2}$$

$$\ddot{x} + \frac{3g}{2a}x - \frac{5}{2}g = 0$$

$$\ddot{x} + \frac{3g}{2a}\left(x - \frac{5a}{3}\right) = 0 \quad ⑤$$

$$\ddot{x} + \omega^2\left(x - \frac{5a}{3}\right) = 0, \text{-------}(1)$$

$$\text{where } \omega = \sqrt{\frac{3g}{2a}}. \quad ⑤$$

$\boxed{20}$

Let $X = x - \dfrac{5a}{3}$.

Then $\dot{X} = \dot{x}$ and $\ddot{x} = \ddot{x}$. ⑤

Now (1) becomes $\ddot{X} + \omega^2 \mathrm{x} = 0$. ⑤

$\boxed{10}$

At the centre $\ddot{\mathrm{X}} = 0$.

$\mathrm{X} = 0$. ⑤

$\therefore x = \dfrac{5a}{3}$. ⑤

$\dot{x} = 0$ when $x = 3a$.

$\therefore$ The amplitade $= 3a - \dfrac{5a}{3} = \dfrac{4a}{3}$. ⑤

The Period $= \dfrac{2\pi}{\omega} \quad = 2\pi\sqrt{\dfrac{2a}{3g}}$. ⑤

$\boxed{20}$

$\dot{X}^2 = \omega^2\left(C^2 - X^2\right)$

The maximum speed occurs when $X = 0$.

$\max \dot{X}^2 = \omega^2 c^2$ ⑤

$\therefore \max \dot{x}^2 = \dfrac{3g}{2a}\cdot\left(\dfrac{4a}{3}\right)^2$

$= \dfrac{3g}{2a}\cdot\dfrac{16a^2}{9} = \dfrac{8}{3}ag$ ⑤

$\therefore$ The maximum speed $= \sqrt{\dfrac{8ag}{3}}$. ⑤

$\boxed{15}$

For the motion under gravity from A to O,

$v^2 = u^2 + 2as$:

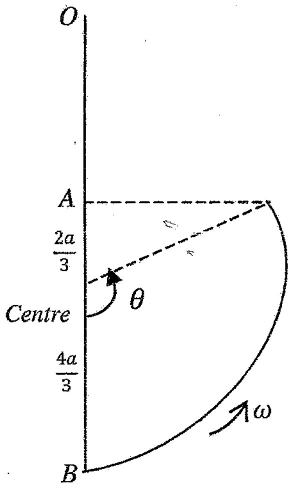
$v^2 = 2ag - 2ga$

$= 0$ ⑤

$\therefore v = 0.$

$\therefore$ P barely reaches O. ⑤

$\boxed{10}$



The time from B to $A = T_1$.

Then $T_1 = \dfrac{\theta}{\omega}$ ⑤

$\theta = \pi - \cos^{-1}\left(\dfrac{1}{2}\right)$ ⑤

$= \pi - \dfrac{\pi}{3}$

$= \dfrac{2\pi}{3}$ ⑤

$\therefore T_1 = \sqrt{\dfrac{2a}{3g}} \cdot \dfrac{2\pi}{3}$ ⑤

The time from A to $O = T_2$.

$\uparrow$ $v = u + at$:

$0 = \sqrt{2ag} - gT_2.$ ⑤

$\therefore T_2 = \sqrt{\dfrac{2a}{g}}$ ⑤

$\therefore$ The total time from B to $O = T_1 + T_2$ ⑤

$$= \sqrt{\frac{2a}{3g}} \cdot \frac{2\pi}{3} + \sqrt{\frac{2a}{g}}$$

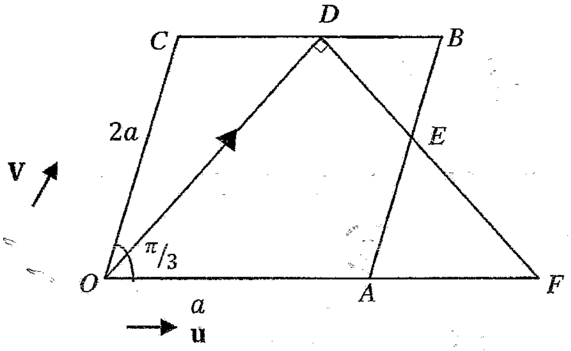
$$= \sqrt{\frac{2a}{27g}}\left(2\pi + 3\sqrt{3}\right).$$ ⑤

$\boxed{40}$

P must be pulled a distance $2a$ from its natured length.

$\boxed{10}$

14.(*a*) Let $OABC$ be a parallelogram with $OA = a$, $OC = 2a$ and $A\hat{O}C = \frac{\pi}{3}$. Also, let $\mathbf{u}$ and $\mathbf{v}$ be the unit vectors in the directions of $\overrightarrow{OA}$ and $\overrightarrow{OC}$ respectively.
Show that $\overrightarrow{OD} = \frac{1}{2}a\mathbf{u} + 2a\mathbf{v}$, where D is the mid-point of BC.
Let E be the point on AB such that OD is perpendicular to DE.
Show that $\overrightarrow{DE} = \frac{a}{2}\mathbf{u} - \frac{a}{3}\mathbf{v}$.
Let F be the point of intersection of the extended lines OA and DE. Show that $\overrightarrow{OF} = \frac{7a}{2}\mathbf{u}$.



$\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD}$ ⑤

$= 2a\underline{v} + \frac{1}{2}a\underline{u}.$ ⑤ + ⑤

$= \frac{1}{2}a\mathbf{u} + 2a\mathbf{v}$ ⑤

$\boxed{20}$

$\overrightarrow{DE} = \overrightarrow{DO} + \overrightarrow{OA} + \overrightarrow{AE}$

$= -\frac{1}{2}a\mathbf{u} - 2a\mathbf{v} + a\mathbf{u} + \lambda\mathbf{v} = \frac{1}{2}a\mathbf{u} + (\lambda - 2a)\mathbf{v}$ ⑩

$\overrightarrow{OD} \perp \overrightarrow{DE} \Rightarrow \overrightarrow{OD}\cdot\overrightarrow{DE} = 0$ ⑤

$\left(\frac{1}{2}a\mathbf{u} + 2a\mathbf{v}\right)\cdot\left(\frac{1}{2}a\mathbf{u} + (\lambda - 2a)\mathbf{v}\right) = 0$ ⑤

$\frac{1}{4}a^2\underbrace{\mathbf{u}\cdot\mathbf{u}}_{=1} + \frac{1}{2}a(\lambda - 2a)\underbrace{\mathbf{u}\cdot\mathbf{v}}_{=\frac{1}{2}} + a^2\underbrace{\mathbf{v}\cdot\mathbf{u}}_{=\frac{1}{2}} + 2a(\lambda - 2a)\underbrace{\mathbf{v}\cdot\mathbf{v}}_{=1} = 0$ ⑤

$\frac{1}{4}a + \frac{1}{4}(\lambda - 2a) + \frac{1}{2}a + 2(\lambda - 2a) = 0$ ⑤

$\frac{9\lambda}{4} = 4a - \frac{3}{4}a + \frac{2a}{4}$

$$\frac{9\lambda}{4}=4a-\frac{a}{4}$$

$$\lambda=\frac{15}{9}a \quad =\frac{5}{3}a.$$ ⑤

$$\therefore \overrightarrow{DE}=\frac{1}{2}a\mathbf{u}+\left(\frac{5a}{3}-2a\right)\mathbf{v}$$

$$=\frac{a}{2}\mathbf{u}-\frac{a}{3}\mathbf{v}$$ ⑤

$\boxed{40}$

$$\overrightarrow{OF}=\overrightarrow{OD}+\overrightarrow{DF}$$ ⑤

⑤ $$\mu\mathbf{u}=\frac{1}{2}a\mathbf{u}+2a\mathbf{v}+\beta\left(\frac{a}{2}\mathbf{u}-\frac{a}{3}\mathbf{v}\right)$$ ⑤

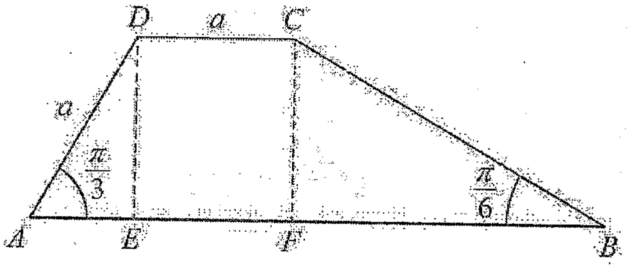
$$\mu=\frac{1}{2}a+\beta\frac{a}{2} \quad \text{and} \quad 0=2a-\frac{\beta a}{3}$$ ⑤ + ⑤

$$\therefore \beta=6 \quad and \quad \mu=\frac{7a}{2}$$

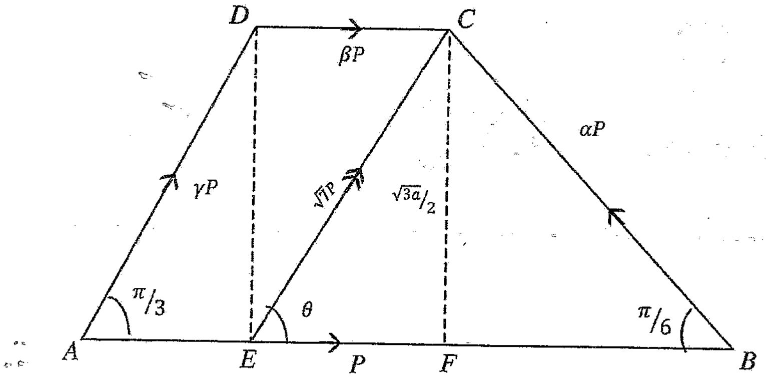
$$\therefore \overrightarrow{OF}=\frac{7a}{2}\mathbf{u}.$$ ⑤

$\boxed{30}$

(b) Let $ABCD$ be a trapezium with AB parallel to DC, $A\hat{B}C=\frac{\pi}{6}$, $B\hat{A}D=\frac{\pi}{3}$ and $AD=DC=a$. The points E and F are on AB such that $A\hat{E}D=A\hat{F}C=\frac{\pi}{2}$ (See the figure). Forces of magnitude P, αP, βP and γP act along AB, BC, DC and AD respectively, in the directions indicated by the order of the letters. It is given that the resultant force of these is of magnitude $\sqrt{7}P$, and it passes through the points E and C in the sense from E to C. Find the values of α, β and γ.



Now, a couple is added to the system such that the line of action of the resultant of the new system passes through the point F. Find the moment of the couple added.

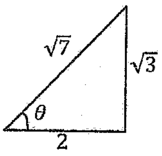


$$\curvearrowright C \quad \left(\frac{\gamma P}{2}+P\right)\frac{\sqrt{3}a}{2}-\frac{\gamma P\sqrt{3}}{2}\left(\frac{a}{2}+a\right)=0$$ ⑩

$$\frac{\gamma}{2}+1-\frac{3\gamma}{2}=0$$

$$r=1.$$ ⑤

$$\uparrow \quad \gamma P\frac{\sqrt{3}}{2}+\alpha P\frac{1}{2}=\sqrt{7}P\cdot\frac{\sqrt{3}}{\sqrt{7}}$$ ⑩



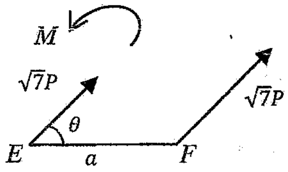
$$\frac{\sqrt{3}}{2}+\frac{\alpha}{2}=\sqrt{3}$$

$$\alpha=\sqrt{3}.$$ ⑤

$$\longrightarrow P+\beta P+P\times\frac{1}{2}-\sqrt{3}P.\frac{\sqrt{3}}{2}=\sqrt{7}P.\frac{2}{\sqrt{7}}$$ ⑩

$$\beta=2.$$ ⑤

$\boxed{45}$



$F \curvearrowright$ $\sqrt{7}Pa\sin\theta - M = 0$ ⑤

$$M = \sqrt{7}Pa \cdot \frac{\sqrt{3}}{\sqrt{7}}$$

$= \sqrt{3}Pa$ ⑤

Moment of the couple $= \sqrt{3}Pa \circlearrowleft$ ⑤

$\boxed{15}$

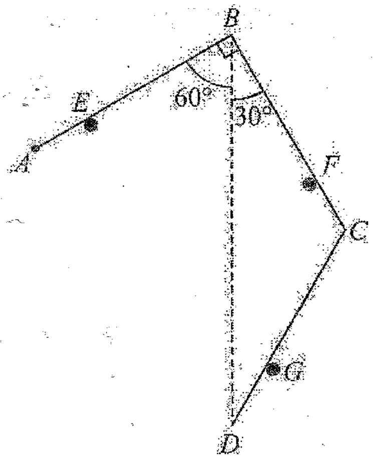
15. (*a*) Three uniform rods AB, BC and CD of equal length $4a$ and equal weight W are smoothly jointed at the end points B and C. The end A is smoothly hinged to a fixed point. The three rods are kept in equilibrium in a vertical plane by placing the rods on three smooth pegs E, F and G such that $AE = CF = DG = a$, $A\hat{B}D = 60^\circ$, $C\hat{B}D = 30^\circ$, and BD is vertical as shown in the figure.

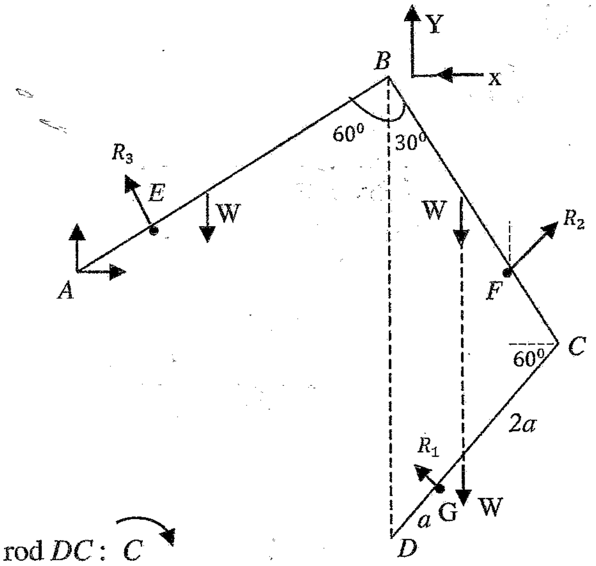
Show that

(i) the magnitude of the reaction exerted on the rod CD by the peg G is $\frac{W}{3}$ and

(ii) the magnitude of the reaction exerted on the rod BC by the peg F is $\frac{11W}{9}$.

Also, find the reaction exerted on the rod BC by the rod AB at the joint B.





⑤

(i) For the rod DC: $\overset{\curvearrowright}{C}$

$$R_1 \times 3a - W \times 2a\cos 60^0 = 0 \quad ⑤$$

$$R_1 = \frac{W}{3}. \quad ⑤$$

$\boxed{15}$

(ii) For BCD: $\overset{\curvearrowleft}{B}$

$$R_2 \times 3a - 2 \times W \times 2a\sin 30^0$$
$$-R_1\cos 30^0 \times 7a\cos 30^0 + R_1\sin 30^0 \times a\sin 30^0 = 0 \quad ⑩$$

$$3R_2 = 2W + \frac{W}{3} \times 7 \times \frac{3}{4} - \frac{W}{3} \times \frac{1}{4}$$

$$= 2W + \frac{20W}{12}$$

$$\therefore R_2 = \frac{11W}{9}. \quad ⑤$$

$\boxed{15}$

For BCD : ⟶

$$R_2 \cos 30^0 - R_1 \cos 30^0 - X = 0 \quad ⑤$$

$$\therefore X = \frac{11W}{9}\frac{\sqrt{3}}{2} - \frac{W}{3}\frac{\sqrt{3}}{2} = \frac{8\sqrt{3}W}{18} = \frac{4\sqrt{3}}{9}W \quad ⑤$$

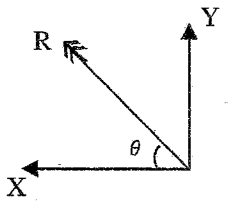
↑ $$Y - 2W + R_1 \sin 30^0 + R_2 \sin 30^0 = 0 \quad ⑤$$

$$\therefore Y = 2W - \frac{W}{3}\times\frac{1}{2} - \frac{11W}{9}\times\frac{1}{2}$$

$$= 2W - \frac{14W}{18}$$

$$= \frac{11W}{9}. \quad ⑤$$

$$R = \sqrt{\frac{48}{81} + \frac{121}{81}}\ W = \sqrt{\frac{169}{81}}\ W = \frac{13}{9}W. \quad ⑤$$



$$\tan\theta = \frac{Y}{X} = \frac{\frac{11W}{9}}{\frac{4\sqrt{3}W}{9}} = \frac{11}{4\sqrt{3}}$$

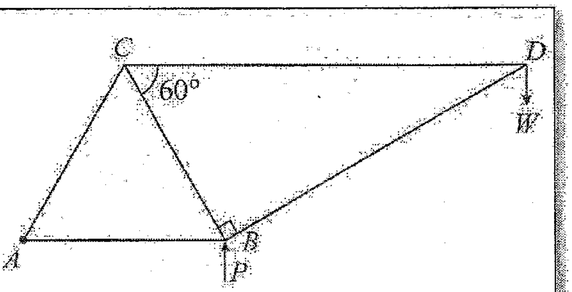
$$\theta = \tan^{-1}\left(\frac{11}{4\sqrt{3}}\right). \quad ⑤$$

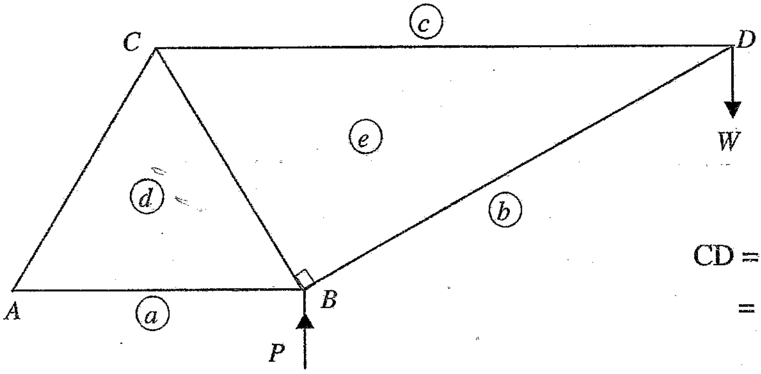
$\boxed{30}$

(*b*) The framework shown in the figure consists of five light rods AB, BC, CA, CD and DB that are smoothly jointed at their ends. It is given that $AB = BC = CA = 2a$, $C\hat{B}D = 90°$ and $B\hat{C}D = 60°$. A load W is suspended at the joint D and the framework is smoothly hinged to a fixed point at A and kept in equilibrium in a vertical plane with AB horizontal, by a force P applied vertically upwards to it at the joint B.

(i) Find the value of P.

(ii) Draw a stress diagram using Bow's notation for the joints D, C and B.
Hence, find the stresses in the rods, stating whether they are tensions or thrusts.





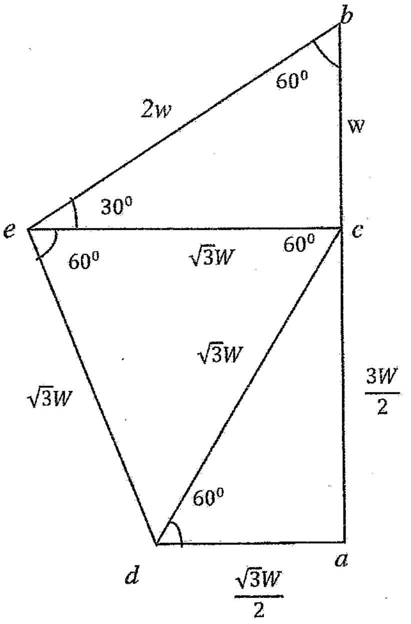
$\text{CD} = 2a\sec 60^0$

$= 4a$

$\overset{\curvearrowright}{\text{A}}$ $\quad P \times 2a - W \times (a + 4a) = 0$ ⑤

$\therefore P = \dfrac{5W}{2}.$ ⑤

$\boxed{10}$



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$\boxed{30}$

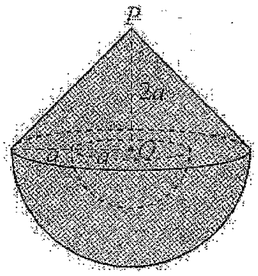
Rod	Magnitude	Tension /Thrust
AB	$\frac{\sqrt{3}W}{2}$	Thrust
BD	$2W$	Thrust
DC	$\sqrt{3}W$	Tension
CA	$\sqrt{3}W$	Tension
BC	$\sqrt{3}W$	Thrust

(50)

$\boxed{50}$

16. Show that the centre of mass of a uniform solid hemisphere of radius a is at a distance $\frac{3}{8}a$ from its centre and the centre of mass of a uniform solid right-circular cone of height h is at a distance $\frac{1}{4}h$ from the centre of its base.

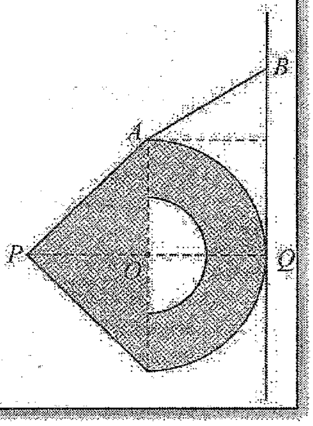
A hemispherical portion of radius a and centre O is carved out from a uniform solid hemisphere of radius $2a$, centre O and density ρ. Now, a uniform solid right circular cone of base radius $2a$ and height $2a$ with density $\lambda\rho$ is rigidly fixed to the remaining portion of the hemisphere, as shown in the adjoining figure. Show that the centre of mass of the body S thus formed, lies at a distance $\frac{(48\lambda+157)}{8(4\lambda+7)}a$ from P, where P is the vertex of the solid cone of S.

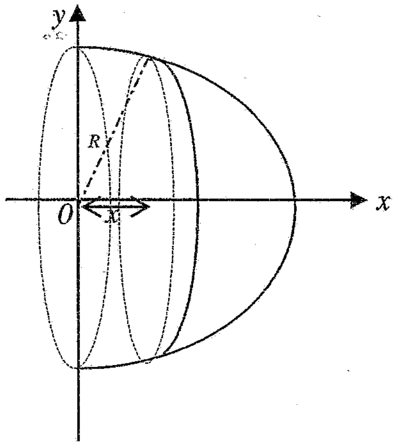


Find the value of λ such that the centre of mass of S, lies at O.

Now, suppose that λ has this value.

Let Q be the point at which the extended line PO meets the outer hemispherical surface of S. Also, let A be a point of the circular edge of S. The body S is kept in equilibrium against a rough vertical wall by means of a light inextensible string with one end attached to the point A and other end to a fixed point B on the vertical wall. In the equilibrium position, the outer hemispherical surface of S touches the wall at the point Q. The points O, A, B, P and Q lie on a vertical plane perpendicular to the wall (see the adjoining figure). Show that $\mu \geq 1$, where μ is the coefficient of friction between the outer hemispherical surface of S and the wall.





By Symmetry the centre of mass G line on the x - axis. ⑤

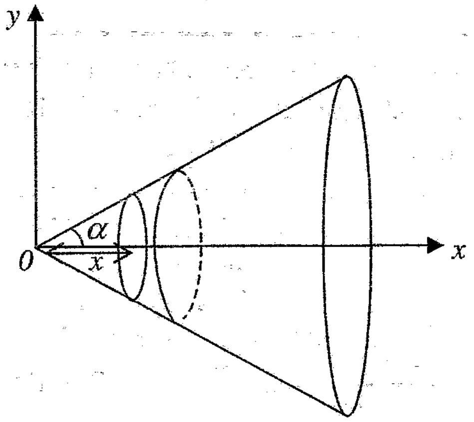
Let $OG = \bar{x}$.

$$\bar{x} = \frac{\int_0^a \pi\left(a^2 - x^2\right)\rho x\mathrm{d}x}{\int_0^a \pi\left(a^2 - x^2\right)\rho\mathrm{d}x}$$ ⑤ + ⑤

$$= \frac{\left(\frac{a^2x^2}{2} - \frac{x^4}{4}\right)\Big|_0^a}{\left(a^2x - \frac{x^3}{3}\right)\Big|_0^a}$$ ⑤ + ⑤

$$= \frac{\frac{a^4}{2} - \frac{a^4}{4}}{a^3 - \frac{a^3}{3}}$$

$$= \frac{3a}{8}$$ ⑤



By symmetry the centre of main G line on the x - axis. ⑤

Let $OG = \overline{x}$

$$\overline{x} = \frac{\int_0^h \pi x^2 \tan^2 \alpha \cdot \rho x dx}{\int_0^h \pi x^2 \tan^2 \alpha \cdot \rho dx}$$ ⑤ + ⑤

$$= \frac{\left.\frac{x^4}{4}\right|_0^h}{\left.\frac{x^3}{3}\right|_0^h}$$ ⑤

$$= \frac{\frac{h^4}{4}}{\frac{h^3}{3}} = \frac{3h}{4}.$$ ⑤

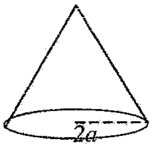
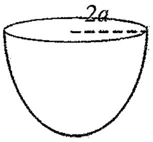
∴ The distance to G from the centre of the base

$$= h - \frac{3h}{4}$$

$$= \frac{h}{4}.$$ ⑤

$\boxed{60}$

By symmetry the centre of main lies on the line joining O and P. ⑤

Object	Mass	Distance to the C.M. from P
	$\frac{1}{3}\pi(2a)^2 2a\lambda\rho$ $=\frac{8}{3}\pi a^3\lambda\rho=4\lambda m$	$\frac{3}{4}(2a)=\frac{3a}{2}$
	$\frac{2}{3}\pi(2a)^3\rho$ $=\frac{16}{3}\pi a^3\rho=8m$	$2a+\frac{3}{8}(2a)$ $=\frac{11a}{4}$
	$\frac{2}{3}\pi a^3\rho=m$	$2a+\frac{3a}{8}$ $=\frac{19}{8}a$
S	$(4\lambda+7)m$	$\bar{x}$

㉟

Where $m=\frac{2}{3}\pi a^3\rho$.

$$4\lambda m\times\frac{3a}{2}+8m\times\frac{11a}{4}-m\times\frac{19a}{8}=(4\lambda+7)m\times\bar{x}$$ ⑩

$$6\lambda a+22a-\frac{19}{8}a=(4\lambda+7)\bar{x}$$

$$\therefore\bar{x}=\frac{(48\lambda+157)}{8(4\lambda+7)}a.$$ ⑤

$\boxed{55}$

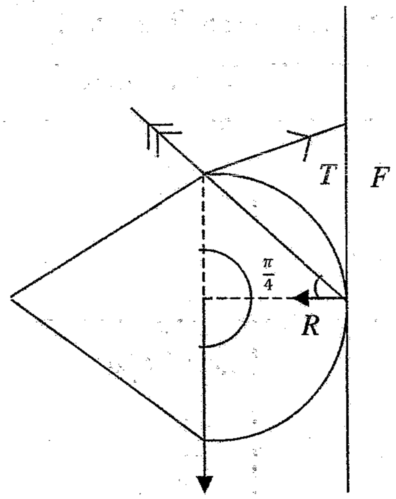
$\bar{x}=2a.$ ⑤

$$\frac{48\lambda+157}{8(4\lambda+7)}a=2a$$ ⑤

$$\therefore 48\lambda+157=64\lambda+112$$

$$\therefore\lambda=\frac{45}{16}.$$ ⑤

$\boxed{15}$



$$\frac{F}{R} = \tan\frac{\pi}{4} = 1 \quad ⑮$$

$$\mu \geq \frac{F}{R} \Rightarrow \mu \geq 1. \quad ⑤$$

$\boxed{20}$

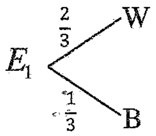
17. (*a*) A box B_1 contains 2 white balls and 3 black balls which are identical in all aspects except for their colours. 3 balls are transferred at random from box B_1 into an empty box B_2. Then a ball is drawn at random from box B_2.
Find the probability that
(i) the ball drawn from box B_2 is white,
(ii) 2 white balls and 1 black ball are transferred from box B_1 into box B_2, given that the drawn ball from box B_2 is white.

Let E_i be the event that i balls out of the 3 balls drawn from B_1 are black ; $i = 1,2,3$.

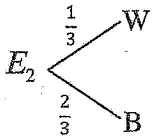
Then, $P(E_1) = \frac{{}^3C_1\,{}^2C_2}{{}^5C_3} = \frac{3}{10}$, ⑩

$P(E_2) = \frac{{}^3C_2\,{}^2C_1}{{}^5C_3} = \frac{6}{10}$, ⑩

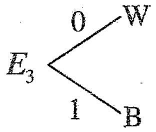
$P(E_3) = \frac{{}^3C_3}{{}^5C_3} = \frac{1}{10}$. ⑩



$P(\mathrm{W}|E_1) = \frac{2}{3}$



$P(\mathrm{W}|E_2) = \frac{1}{3}$ ⑩



$P(\mathrm{W}|E_3) = 0$

Total Prabability Law :

$$P(\mathrm{W}) = P(\mathrm{W}|E_1)P(E_1) + P(\mathrm{W}|E_2)P(E_2) + P(\mathrm{W}|E_3)P(E_3)$$

$$= \frac{2}{3}\times\frac{3}{10} + \frac{1}{3}\times\frac{6}{10} + 0\times\frac{1}{10}$$ ⑩

$$= \frac{2}{5}.$$ ⑤

$\boxed{55}$

Baye's Theorem :

$$P(E_1|\mathrm{W}) = \frac{P(\mathrm{W}|E_1).P(E_1)}{P(\mathrm{W})}$$ ⑩

$$= \frac{\frac{2}{3}\times\frac{3}{10}}{\frac{2}{5}}$$

$$= \frac{1}{2}$$ ⑤

$\boxed{15}$

(*b*) The times taken to solve a puzzle by 20 students were coded by subtracting 10 from each of the times and then dividing by 2.

The frequency distribution of the coded data with 2 missing frequencies is given below:

Coded times (in minutes)	**frequency**
0 – 2	2
2 – 4	f_1
4 – 6	9
6 – 8	f_2
8 – 10	1

Estimated mean for the coded times is given to be 4.4 minutes. Show that $f_1 = 6$ and $f_2 = 2$. Estimate the standard deviation and the mode of the coded times.

Now, estimate the mean, the standard deviation and the mode of the actual times taken to solve the puzzle.

(*b*)

Coded time (in minutes)	f_i	x_i	$f_i x_i$	$f_i x_i^2$
0 - 2	2	1	2	2
2 - 4	f_1	3	$3f_1$	$9f_1 = 54$
4 - 6	9	5	45	225
6 - 8	f_2	7	$7f_2$	$49f_2 = 98$
8 - 10	1	9	9	81
			$56+3f_1+7f_2$	460

$$12 + f_1 + f_2 = 20$$

$$\therefore \quad f_1 + f_2 = 8 \quad \text{-----------(1)}$$ ⑤

$$\frac{\sum_{i=1}^{5} f_i x_i}{\sum_{i=1}^{5} f_2} = \frac{56 + 3f_1 + 7f_2}{20} = 4.4$$ ⑩

$$\therefore 3f_1 + 7f_2 = 32 \quad \text{-----------(2)}$$ ⑤

$$(1),(2) \Rightarrow f_1 = 6 \quad \text{and} \quad f_2 = 2$$ ⑤

$\boxed{25}$

$$\text{Standard deviation} = \sqrt{\frac{460}{20} - (4.4)^2}$$ ⑩ + ⑤

for 460

$$= \sqrt{3.64}$$ ⑤

$$\approx 1.91$$

Model Class : 4 – 6

$$\text{Mode } = L + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right)c$$

$$= 4 + \left(\frac{9-6}{(9-6)+(9-2)}\right) \times 2$$ ⑤

$$= 4 + \left(\frac{3}{3+7}\right) \times 2$$

$$= 4.6$$ ⑤

$\boxed{30}$

Let the actual times be y.

Then, $x = \frac{y-10}{2}$.

$\therefore\ y = 2x + 10.$ ⑤

$\bar{y} = 2\bar{x} + 10$

$\therefore\ \bar{y} = 2\times 4.4 + 10$

Actual mean $= 18.8$ ⑤

$\sigma_y = 2\sigma_x$ ⑤

Actual s.d. $= 2\times\sqrt{3.64}$ $\approx 2\times 1.91$ $= 3.82$ ⑤

Actual mode $= 2\times 4.6 + 10$

$= 19.2$ ⑤

$\boxed{25}$