## G. C. E. (Advanced Level) Examination, August 2011
## Combined Mathematics II
### Scheme of Marking
### Part A

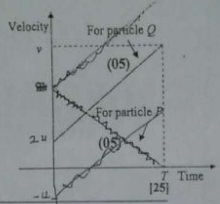
1. A particle $P$ is projected vertically upwards from a point $O$ in space with velocity $2u$. At the same instant, another particle $Q$ is projected vertically downwards from the same point $O$ with the velocity $u$. Both particles move under gravity. Draw the velocity-time graphs for the motions of the particles $P$ and $Q$ **in the same figure** and show that the speed of the particle $Q$ when the particle $P$ reaches its maximum height, is $3u$.

Let $T$ be the time required for the particle $P$ to reach the maximum height.
Let $v$ be the required velocity of the particle $Q$.

Then $\dfrac{2u}{T} = g \rightarrow (1)$ (05)

Also, $\dfrac{v-u}{T} = g \rightarrow (2)$ (05)

From (1) and (2) we get
$v - u = 2u \Rightarrow v = 3u$ (05)

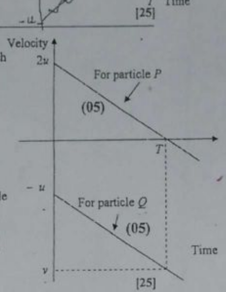



**Aliter:**
Let $T$ be the time required for the particle $P$ to reach the maximum height.
Let $v$ be the required velocity of the particle $Q$.

Then $\dfrac{2u}{T} = g \rightarrow (1)$ (05)

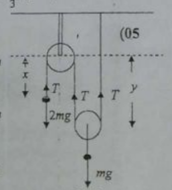
Also, $\dfrac{-u-V}{T} = g \rightarrow (2)$ (05)

From (1) and (2) we get
$-u - v = 2u \Rightarrow v = -3u$ (05)
Thus, the speed of the particle $Q$ when the particle reaches the maximum height is $3u$.




2. One end of a light inextensible string which passes over a smooth fixed pulley carries a particle of mass $2m$. The string passes under a smooth light pulley which carries a particle of mass $m$. The other end of the string is attached to a ceiling as shown in the figure. The system moves freely under gravity. Show that the tension of the string is $\frac{2}{3}mg$.

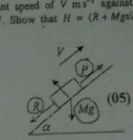
Since the string is inextensible $x + 2y =$ constant.
$\therefore \ddot{x} + 2\ddot{y} = 0 \rightarrow (1)$ (05)
Applying $\overset{F=ma}{\cancel{P=mf}}$ for the motion of the particle of mass $2m$ vertically upwards we have
$T - 2mg = 2m(-\ddot{x}) \rightarrow (2)$ (05)
Applying $\overset{F=ma}{\cancel{P=mf}}$ for the motion of the particle of mass $m$ vertically upwards we have
$2T - mg = m(-\ddot{y}) \rightarrow (3)$ (05)
$(2) + 4\times(3) \Rightarrow 9T - 6mg = -2m(\ddot{x} + 2\ddot{y}) = 0$ Form (1) we have
$T = \frac{2}{3}mg$. (05)




3. The total mass of a cyclist and his bicycle is $M$ kg. When he rides directly up a straight road inclined at an angle $\alpha$ to the horizontal, at a constant speed of $V$ m s$^{-1}$ against a resistance to motion of $R$ N, he works at a constant rate of $H$ W. Show that $H = (R + Mg\sin\alpha)V$.

Applying $\overset{F=mg}{\cancel{P=mf}}$ for the motion of the cyclist along the road upwards we have
$P - R - Mg\sin\alpha = 0$ (10)/0
Also, we have
$H = PV$ (05)

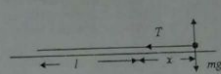
$\therefore \dfrac{H}{V} = R + Mg\sin\alpha \Rightarrow H = (R + Mg\sin\alpha)V$. (05)




[25]

4. A thin light elastic spring of natural length $l$ and modulus of elasticity $\lambda$ rests on a smooth horizontal table. One of its ends is fasten to a fixed point on the table. A particle of mass $m$ is attached to the other end. The spring is stretched along the table and released. Show that the particle performs a simple harmonic motion with periodic time $2\pi\sqrt{\frac{ml}{\lambda}}$.

$T = \lambda\dfrac{x}{l}$ (05)
Applying $\overset{F=ma}{\cancel{P=mf}}$ for the motion of the particle of mass $m$ horizontally we have
$-T = m\ddot{x}$ (05)
$\therefore \ddot{x} = -\dfrac{\lambda}{ml}x$ (05)
Thus, the particle performs a simple harmonic motion. (05)
The periodic time
$= \dfrac{2\pi}{\sqrt{\lambda/ml}} = 2\pi\sqrt{\dfrac{ml}{\lambda}}$. (05)

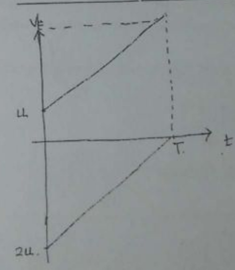



[25]

5. Let $-2\mathbf{p} + 5\mathbf{q}$, $7\mathbf{p} - \mathbf{q}$ and $\mathbf{p} + 3\mathbf{q}$ be the position vectors of three points $A$, $B$ and $C$ respectively, with respect to a fixed origin $O$, where $\mathbf{p}$ and $\mathbf{q}$ are two non-parallel vectors. Show that the points $A$, $B$ and $C$ are collinear and find the ratio in which $C$ divides $AB$.

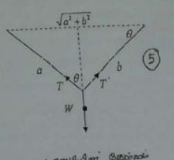
$\overrightarrow{AC} = \mathbf{p} + 3\mathbf{q} - (-2\mathbf{p} + 5\mathbf{q}) = 3\mathbf{p} - 2\mathbf{q}$. (05)
$\overrightarrow{CB} = 7\mathbf{p} - \mathbf{q} - (\mathbf{p} + 3\mathbf{q}) = 6\mathbf{p} - 4\mathbf{q} = 2\overrightarrow{AC}$. (05) (05)
$\overrightarrow{AC} = \frac{1}{2}\overrightarrow{CB}$ (05)

Thus, the points $A$, $B$ and $C$ are collinear and $\cancel{AC = \frac{1}{2}CB}$ $\longrightarrow \begin{cases} AC : CB = 1 : 2 \\ \frac{AC}{CB} = \frac{1}{2} \end{cases}$ [25]
(05) (05)




$\Rightarrow V = 3u$

6. A weight $W$ is suspended by two light inextensible strings of lengths $a$ and $b$ from two points at the same horizontal level which are at a distance $\sqrt{a^2+b^2}$ apart. Show that the tensions in the strings are $\frac{Wa}{\sqrt{a^2+b^2}}$ and $\frac{Wb}{\sqrt{a^2+b^2}}$.




Resolving vertically we have

$T\cos\theta + T'\sin\theta = W$ (05)

$T\frac{b}{\sqrt{a^2+b^2}} + T'\frac{a}{\sqrt{a^2+b^2}} = W$ (05)

$Tb + T'a = W\sqrt{a^2+b^2} \rightarrow (1)$

Resolving horizontally we have

$T\sin\theta - T'\cos\theta = 0$ (05)

$T\frac{a}{\sqrt{a^2+b^2}} - T'\frac{b}{\sqrt{a^2+b^2}} = 0$

$Ta - T'b = \cancel{W\sqrt{a^2+b^2}}\ 0 \rightarrow (2)$

$(1)\times b + (2)\times a \Rightarrow T = \frac{Wa}{\sqrt{a^2+b^2}}$. (05)

$(1)\times a - (2)\times b \Rightarrow T' = \frac{Wb}{\sqrt{a^2+b^2}}$ (05) [25]

---

7. Let $A$ and $B$ be two exhaustive events in a sample space $\Omega$ (that is $A\cup B = \Omega$). If $P(A)=\frac{2}{5}$ and $P(A\cap B)=\frac{1}{3}$, find (i) $P(B)$, (ii) $P(A|B)$, (iii) $P(A'|B')$, where $A'$ and $B'$ are the complementary events of $A$ and $B$ respectively.

(i) $P(\Omega) = P(A\cup B) = P(A) + P(B) - P(A\cap B)$.

$1 = \frac{2}{5} + P(B) - \frac{1}{3} \Rightarrow P(B) = \frac{14}{15}$.

(05) (05)

(ii) $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{\frac{1}{3}}{\frac{14}{15}} = \frac{5}{14}$. (05)

(iii) $P(A'|B') = \frac{P(A'\cap B')}{P(B')} = \frac{P\{(A\cup B)'\}}{P(B')} = \frac{P(\Phi)}{P(B')} = 0 \quad \because P(\Phi) = 0$

(05) (05) [25]

---

8. Two friends attempt independently to solve a problem; their probabilities of success being $\frac{1}{3}$ and $\frac{1}{4}$. Find the probability that (i) both of them, (ii) none of them, will succeed in solving the problem.

$P(A)=\frac{1}{3}$ and $P(B)=\frac{1}{4}$, where $A$ and $B$ are the events that the two friends being success in solving the problem.

(i) $P(A\cap B) = P(A)P(B) = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right) = \frac{1}{12}$.

(05) (05)

(ii) $P\{(A\cup B)'\} = 1 - \{P(A) + P(B) - P(A\cap B)\} = 1 - \left\{\frac{1}{3} + \frac{1}{4} - \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)\right\} = 1 - \frac{1}{2} = \frac{1}{2}$

(05) (05) (05) [25]



9. The daily expenditure of 1000 families is given in the following table:

| Daily expenditure in Rupees | 400 - 600 | 600 - 800 | 800 - 1000 | 1000 - 1200 | 1200 - 1400 |
|---|---|---|---|---|---|
| Number of families | 50 | $x$ | 500 | $y$ | 50 |

If the median of the distribution is 900 Rupees, find the frequencies $x$ and $y$, and show that the mean of the distribution is also 900 Rupees.

Since the median is 900 we have

$50 + x + \frac{500}{200}\times 100 = 500 \Rightarrow x = 200$, $50 + y + \frac{500}{200}\times 100 = 500 \Rightarrow y = 200$ 1000 equal

(05) (05) (05) (05)

Since the distribution is symmetrical the mean is equal to the median.

Thus, the mean is also 900. (05) [25]

---

10. Over the past 15 months, the number of orders received for a certain product has an average of 24 orders per month. The best three months has an average of 35 orders per month. There were 11, 14, 16 and 22 orders for the product in the lowest four months.

Find (i) the average of the number of orders received in the remaining 8 months,

(ii) the first quartile of the number of orders of the 15 months.

(i) Total sum $= 24\times 15 = 360$ 5

Total sum of the best three months $= 35\times 3 = 105$

Total sum of the lowest four months $= 11+14+16+22 = 63$

Total of the remaining eight months $= 360 - 105 - 63 = 192$ (05).

Average of the remaining eight months $= \frac{192}{8} = 24$.

~~(05)~~ (05) [15]

(ii) Since there are 15 data fourth datum should be the first quartile of the distribution. (05)

Thus, 22 is the first quartile of the distribution. (05) [10]

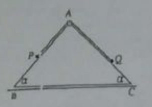
## Part B

11 (a) The top-most points $A$, $B$ and $C$ of three lamp-posts lie in a horizontal plane at the vertices of an equilateral triangle of side $a$. A wind blows in the direction of $\overrightarrow{AC}$ at a steady speed $u$. A bird, whose speed relative to the wind is $v\ (> u)$, flies from $A$ to $B$ along $AB$ and then from $B$ to $C$ along $BC$.

Draw the velocity triangles of relative velocities for both parts of the journey **in the same figure**.

**Hence**, show that the total time taken for the journey from $A$ to $C$ through $B$ is $\dfrac{4a}{u+\sqrt{4v^2-3u^2}}$.

(b) A small smooth pulley is fixed at the vertex $A$ of the triangular vertical cross-section $ABC$ of a smooth wedge of mass $2m$ through its centre of mass. The face through $BC$ is placed on a fixed smooth horizontal table. It is given that $AB$ and $AC$ are lines of greatest slope of the relevant faces and $A\hat{B}C = A\hat{C}B = \alpha$. Two smooth particles $P$ and $Q$ of masses $m$ and $\lambda m\ (\lambda > 1)$ respectively, are attached to the ends of a light inextensible string. The string passes over the pulley and the particles $P$ and $Q$ are placed on $AB$ and $AC$ respectively, with the string taut as shown in the figure.




The system is released from rest.

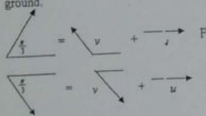
Obtain the equations of motion for the particles $P$ and $Q$ along $BA$ and $AC$ respectively, and for the system horizontally.

Show that the magnitude of the acceleration of each of the particles $P$ and $Q$ relative to the wedge is

$$\frac{(\lambda-1)(\lambda+3)g\sin\alpha}{(\lambda+1)[(\lambda+3)-(\lambda+1)\cos^2\alpha]}$$

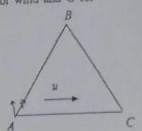
When the particle $Q$ reaches $C$, the string is suddenly broken. Assuming that $P$ has not reached the pulley, write down the magnitude of the acceleration of the particle $P$ relative to the wedge just after the string is broken.

(a) Vel. B, G = Vel. B, W + Vel W, G , where B for bird, W for wind and G for ground.




For motion along $AB$

For motion along $BC$

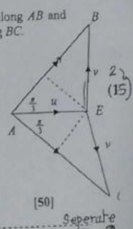



$AEB$ is the velocity triangles of relative velocities for the motion along $AB$ and $AEC$ is the velocity triangles of relative velocities for the motion along $BC$.

The velocity of the bird relative to ground along $AB$ is

$$\sqrt{v^2-u^2\sin^2\frac{\pi}{3}}+u\cos\frac{\pi}{3}=\sqrt{v^2-\frac{3}{4}u^2}+\frac{1}{2}u=\frac{1}{2}\left\{\sqrt{4v^2-3u^2}+u\right\}$$

(10) (05)




2
(15)
5

Time taken to fly $AB$ is $\dfrac{2a}{u+\sqrt{4v^2-3u^2}}$. (05)

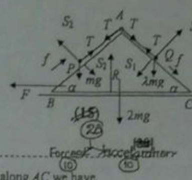
By symmetry the time taken to fly $BC$ is $\dfrac{2a}{u+\sqrt{4v^2-3u^2}}$ (10)

Thus the total time taken to fly $AB$ and $BC$ is $\dfrac{4a}{u+\sqrt{4v^2-3u^2}}$ (05) [50]

Seperate
5



(b) Let the acceleration of the particle $P$ relative to the wedge be $f$ along $BA$.
Then the acceleration of the particle $Q$ relative to the wedge is $f$ along $AC$.
Let the acceleration of the wedge be $F$ alone $CA$.




~~(15)~~ (25)
Forces (10) Acceleration (10)

AB → (10)

35

CA → (10)

Applying ~~$P = mf$~~ $F = ma$ for the motion of the particle $P$ along $BA$ we have
$-mg\sin\alpha + T = m(f - F\cos\alpha) \rightarrow (1)$ ~~(15)~~ 0

Applying ~~$P = mf$~~ $F = ma$ for the motion of the particle $Q$ along $AC$ we have
$\lambda mg\sin\alpha - T = \lambda m(f - F\cos\alpha) \rightarrow (2)$ ~~(15)~~ 0 [15]

Applying ~~$P = mf$~~ $F = ma$ for the motion of the system horizontally along $CB$ we have
$0 = 2mF + m(F - f\cos\alpha) + \lambda m(F - f\cos\alpha) \rightarrow (3)$ ~~(15)~~ 0 [15]

$\therefore\ F = \dfrac{1+\lambda}{3+\lambda} f\cos\alpha$. (05)

$$(1)+(2) \Rightarrow -g(1-\lambda)\sin\alpha = (1+\lambda)f - (1+\lambda)F\cos\alpha \quad (05)$$

$$= (1+\lambda)f\left\{1-\frac{(1+\lambda)}{3+\lambda}\cos^2\alpha\right\} \quad (05)$$

$$= \frac{(1+\lambda)}{(3+\lambda)}\{(3+\lambda)-(1+\lambda)\cos^2\alpha\}f$$

Thus we have $f = \dfrac{(\lambda-1)(3+\lambda)g\sin\alpha}{(1+\lambda)\{(3+\lambda)-(1+\lambda)\cos^2\alpha\}}$. (05)

i.e. the magnitude of the acceleration of the particle $P$ or $Q$ relative to the wedge is

$\dfrac{(1-\lambda)(3+\lambda)g\sin\alpha}{(1+\lambda)\{(3+\lambda)-(1+\lambda)\cos^2\alpha\}}$. ~~(05)~~ ~~[25]~~ 20

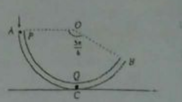
The magnitude of the acceleration of the particle $P$ relative to the wedge just after the string is broken can be found by setting $\lambda = 0$ in

or T=0

$f = \dfrac{(1-\lambda)(3+\lambda)g\sin\alpha}{(1+\lambda)\{(3+\lambda)-(1+\lambda)\cos^2\alpha\}}$. (10)

Thus we have the magnitude of the acceleration of the particle $P$ relative to the wedge just after the string is broken as $f_1 = \dfrac{3g\sin\alpha}{3-\cos^2\alpha}$ ~~(05)~~ (05) [15]

12. A thin smooth tube $ACB$ in the shape of a circular arc of radius $a$ that subtends an angle $\frac{5\pi}{5}$ at its centre $O$ is fixed in a vertical plane with $OA$ horizontal and the lowest point $C$ of the tube touching a fixed horizontal floor as shown in the figure. A smooth particle $P$ of mass $m$ is projected vertically downwards into the tube at the end $A$ with speed $\sqrt{2ga}$.




Show that the speed of the particle $P$, when $OP$ makes an angle $\theta \left(0 \le \theta \le \frac{\pi}{2}\right)$ with $OA$ is $\sqrt{2ga(1+\sin\theta)}$ and the magnitude of the reaction on the particle P from the tube is $mg(2+3\sin\theta)$.

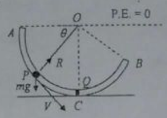
The particle $P$, when it reaches the point $C$, strikes another smooth particle $Q$ of mass $m$ which is at rest inside the tube at $C$. The coefficient of resitution between the particles $P$ and $Q$ is $\frac{1}{2}$.

Find the speed of the particle $P$ just before the collision and show that the speeds of the particles $P$ and $Q$ just after the collision are $\frac{1}{2}\sqrt{ga}$ and $\frac{3}{2}\sqrt{ga}$ respectively.

Show further that the particle $P$ never leaves the tube and that the particle $Q$ reaches the point $B$ with speed $\frac{1}{2}\sqrt{5ga}$.

Find the maximum height from the floor reached by the particle $Q$ after it leaves the tube.

The reaction $R$ is perpendicular to the direction of motion and so it does no work.
Thus, by the conservation of energy for the system we have




$\frac{1}{2}mV^2 - mga\sin\theta = \frac{1}{2}m(2ga)$ ~~mga~~

(15) / 0

i.e. $V^2 = 2ga(1+\sin\theta)$.

$\therefore V = \sqrt{2ga(1+\sin\theta)}$. (05)

i.e. the speed of the particle $P$, when $OP$ makes an angle $\theta$ with $OA$, is $\sqrt{2ga(1+\sin\theta)}$.

~~[20]~~ [20]

Applying $P = mf$ for the motion of the particle $P$ along $PO$ we have

$R - mg\sin\theta = m\frac{V^2}{a} = m\,2ga(1+\sin\theta)$. (15)

(10) $\therefore R = mg(2+3\sin\theta)$. (05) (5)

i.e. the magnitude of the reaction on the particle $P$ from the tube is $mg(2+3\sin\theta)$.

[20]

Let $V_1$ be the velocity of the particle $P$ when it reaches the point $C$.

Then taking $\theta = \frac{\pi}{2}$ in $V = \sqrt{2ga(1+\sin\theta)}$ we have

$V_1 = \sqrt{2ga\left(1+\sin\frac{\pi}{2}\right)} = 2\sqrt{ga}$. (10) [10]



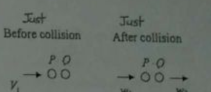
Momentum conservation:

$m2\sqrt{ga} = mw_2 + mw_1$ (10)

i.e. $w_2 + w_1 = 2\sqrt{ga} \rightarrow (1)$

Newton's law of restitution:

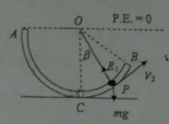
$-(w_1 - w_2) = eV_1 = \frac{1}{2}.2\sqrt{ga} = \sqrt{ga} \rightarrow (2)$ (10)




$(1)+(2) \Rightarrow w_2 = \frac{3\sqrt{ga}}{2}$ (05) and $(1)-(2) \Rightarrow w_1 = \frac{\sqrt{ga}}{2}$. (05)

[30]

The reaction $R_1$ is perpendicular to the direction of motion and so it does no work. Thus, by the conservation of energy for the particle $P$ we have




$\frac{1}{2}mV_2^2 - mga\cos\beta = \frac{1}{2}m\,w_1^2 - mga$. ~~(10)~~ (15) / 0

$\frac{1}{2}mV_2^2 - mga\cos\beta = \frac{1}{2}m\left(\frac{ga}{4}\right) - mga$

i.e. $V_2^2 = 2ga\left(\cos\beta - \frac{7}{8}\right)$. (05)

$V_2 = 0 \Leftrightarrow \cos\beta = \frac{7}{8} > \cos\frac{\pi}{3} \Rightarrow \beta < \frac{\pi}{3} \quad \because 0 \le \beta < \frac{\pi}{2}$

(05) (05) Important

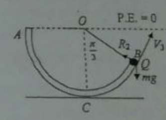
Let this angle be $\beta_0$.

Thus the particle $P$ oscillates between $-\beta_0$ and $\beta_0$. ~~(05)~~

Hence, the particle $P$ never leaves the tube. (05)

[35]

The reaction $R_2$ is perpendicular to the direction of motion and so it does no work. Thus, by the conservation of energy for the particle $Q$ when it reaches the point $B$, we have




$\frac{1}{2}mV_3^2 - mga\cos\frac{\pi}{3} = \frac{1}{2}m\,w_2^2 - mga$. (15) / 0

$\frac{1}{2}mV_3^2 - \frac{1}{2}mga = \frac{1}{2}m\left(\frac{9ga}{4}\right) - mga$

i.e. $V_3^2 = \frac{5ga}{4} \Rightarrow V_3 = \frac{1}{2}\sqrt{5ga}$. (05)

Thus the particle $Q$ reaches the point $B$ with speed $\frac{1}{2}\sqrt{5ga}$. [20]

Applying $v^2 = u^2 + 2fs$ vertically form $B$ to the maximum point it reaches for the motion of the particle $Q$ we have

(10) $0 = V_3^2 \sin^2\frac{\pi}{3} - 2gs = \left(\frac{5ga}{4}\right)\left(\frac{3}{4}\right) - 2gs$, where $s$ is the maximum height the particle $Q$ reaches in its vertical motion.

$s = \frac{15}{32}a$. (05)

maximum hight From the base $= \frac{a}{2} + \frac{15a}{32}$ [15]

$= \frac{31a}{32}$.

13. A particle $P$ of mass $m$ is attached to one end of a light elastic string of natural length $l$. The other end of the string is attached to a fixed point $O$ at a height $4l$ from a horizontal floor. When the particle $P$ hangs in equilibrium, the extension of the string is $l$.

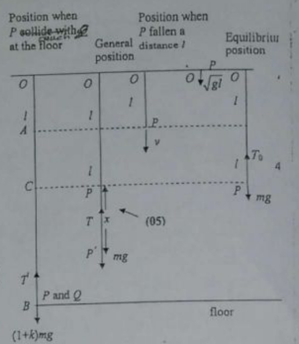
Show that the modulus of elasticity of the string is $mg$.

The particle $P$ is now held at $O$ and projected vertically downwards with a velocity $\sqrt{gl}$. Find the velocity of the particle $P$ when it has fallen a distance $l$.

Write down the equation of motion for the particle $P$, when the length of the string is $2l+x$, where $-l \le x \le 2l$, and show that $\ddot{x}+\frac{g}{l}x=0$, in the usual notation.

Assuming that the above equation gives $\dot{x}^2=\frac{g}{l}(c^2-x^2)$, where $c\,(>0)$ is a constant, find $c$.

Show that the particle $P$ comes to instantaneous rest when it reaches the floor and that the time taken from $O$ to reach the floor is $\frac{1}{3}\left(3\sqrt{3}-3+2\pi\right)\sqrt{\frac{l}{g}}$.




Let $T_0$ be the tension in the string in equilibrium position.

Then $T_0=\frac{\lambda l}{l}$ by Hooke's law.

(10)

But $T_0=mg$. (10)

Thus we have $\lambda=mg$. (05) [25]

Applying $v^2=u^2+2fs$ vertically downwards to the particle $P$ we have

$v^2=gl+2g(l)=3gl$, (10)

where $v$ is the velocity of $P$ when it has fallen a distance $l$.

Thus we have $v=\sqrt{3gl}$. (05) ~~[20]~~ $\boxed{15}$

Again by Hooke's law we have

$T=\frac{mg(l+x)}{l}$. (10)

Applying Newton's law vertically downwards for the particle $P$ we have

$mg-T=m\ddot{x}$. (10)

i.e. $mg-\frac{mg(l+x)}{l}=m\ddot{x} \Rightarrow \ddot{x}+\frac{g}{l}x=0$

(05) (05)

ısLió (5)

~~[30]~~ $\boxed{35}$

any other mtd.

$\dot{x}=v=\sqrt{3gl}$, when $x=-l$. (10)

Thus from $\dot{x}^2=\frac{g}{l}\left(c^2-x^2\right)$ we have

$3gl=\frac{g}{l}\left(c^2-l^2\right) \Rightarrow c=2l$

(05) (05) [20]

$\therefore\ \dot{x}^2=\frac{g}{l}\left(4l^2-x^2\right)$.

$\dot{x}>0$ for $-l\le x<2l$ and $\dot{x}=0$ for $x=2l$.

(05) (05)

Thus the particle $P$ comes to instantaneous rest when it reaches the floor. (05) [15]



Let $t_1$ be time taken for the particle $P$ to fall under gravity from $O$ to $A$.

Then from $v=u+ft$ we have

$\sqrt{3gl}=\sqrt{gl}+gt_1 \Rightarrow t_1=\left(\sqrt{3}-1\right)\sqrt{\frac{l}{g}}$

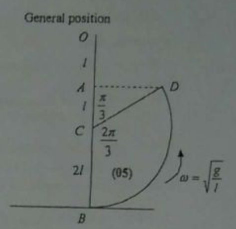
(10) (05)

Let $t_2$ be time taken for the particle $P$ to move in simple harmonic motion from $A$ to $B$.

Then from the figure just above we have

$\sqrt{\frac{g}{l}}\,t_2=\frac{2\pi}{3} \Rightarrow t_2=\frac{2\pi}{3}\sqrt{\frac{l}{g}}$

(10) (05)




$t_1+t_2=\left(\sqrt{3}-1\right)\sqrt{\frac{l}{g}}+\frac{2\pi}{3}\sqrt{\frac{l}{g}}=\frac{1}{3}\left(3\sqrt{3}-3+2\pi\right)\sqrt{\frac{l}{g}}$. (05)

Thus the time taken from $O$ to reach the floor is $\frac{1}{3}\left(3\sqrt{3}-3+2\pi\right)\sqrt{\frac{l}{g}}$. [40]

14. (a) Define the dot product $\mathbf{a}\cdot\mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$.

Assuming $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{c}+\mathbf{d}) = \mathbf{a}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{c}+\mathbf{a}\cdot\mathbf{d}+\mathbf{b}\cdot\mathbf{d}$ for any four vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$, show that $|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + 2(\mathbf{a}\cdot\mathbf{b}) + |\mathbf{b}|^2$.

Write down a similar expression for $|\mathbf{a}-\mathbf{b}|^2$.

Show that, if $|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}-\mathbf{b}|^2$ then $\mathbf{a}\cdot\mathbf{b} = 0$.

Hence, show that if the diagonals of a parallelogram are equal, then it is a rectangle.

(b) The points $A$, $B$, $C$, $D$, $E$ and $F$ are the vertices of a regular hexagon of side $2a$ metres taken in the anti-clockwise sense. Forces of magnitude $P$, $2P$, $3P$, $4P$, $5P$, $L$, $M$ and $N$ newtons act along $AB$, $CA$, $FC$, $DF$, $ED$, $BC$, $FA$ and $FE$ respectively, in the sense indicated by the order of the letters.

If the system is in equilibrium, find $L$, $M$ and $N$ in terms of $P$.

(a) $\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$, where $\theta$ is the angle between the two non zero vectors $\mathbf{a}$ and $\mathbf{b}$. (10) [10]

o/w. it is defined to be zero

Taking $\mathbf{c} = \mathbf{a}$ and $\mathbf{d} = \mathbf{b}$ in $(\mathbf{a}+\mathbf{b}).(\mathbf{c}+\mathbf{d}) = \mathbf{a}.\mathbf{c}+\mathbf{b}.\mathbf{c}+\mathbf{a}.\mathbf{d}+\mathbf{b}.\mathbf{d}$ we have

$(\mathbf{a}+\mathbf{b}).(\mathbf{a}+\mathbf{b}) = \mathbf{a}.\mathbf{a}+\mathbf{b}.\mathbf{a}+\mathbf{a}.\mathbf{b}+\mathbf{b}.\mathbf{b}$ (10)

But $(\mathbf{a}+\mathbf{b}).(\mathbf{a}+\mathbf{b}) = |\mathbf{a}+\mathbf{b}||\mathbf{a}+\mathbf{b}|$, $\mathbf{a}.\mathbf{a} = |\mathbf{a}||\mathbf{a}|$, $\mathbf{b}.\mathbf{b} = |\mathbf{b}||\mathbf{b}|$ and $\mathbf{a}.\mathbf{b} = \mathbf{b}.\mathbf{a}$

(05) (05) (05)

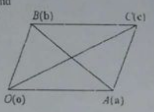
$|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + 2(\mathbf{a}.\mathbf{b}) + |\mathbf{b}|^2$. (05) [30]

$|\mathbf{a}-\mathbf{b}|^2 = |\mathbf{a}|^2 - 2(\mathbf{a}.\mathbf{b}) + |\mathbf{b}|^2$. (10) [10]

$|\mathbf{a}+\mathbf{b}|^2 - |\mathbf{a}-\mathbf{b}|^2 = 4\,\mathbf{a}.\mathbf{b}$ (05)

If $|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}-\mathbf{b}|^2$ then $\mathbf{a}.\mathbf{b} = 0$. (05) [10]

Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be the position vectors of the vertices $A$, $B$ and $C$ of a parallelogram $OACB$ with respect to the point $O$. (05)




$OC = |\mathbf{c}|$ and $AB = |\mathbf{b}-\mathbf{a}|$.

But $\mathbf{c} = \mathbf{a}+\mathbf{b}$

Thus, we have $OC = |\mathbf{a}+\mathbf{b}|$

$OC = AB \Rightarrow |\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}|$. (05)

From the above result proved we have $\mathbf{a}.\mathbf{b} = 0$. (05)

This means that $OA$ is perpendicular to $OB$. (05)

Thus $OACB$ is a rectangle. [20]

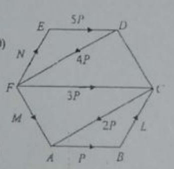
(b) Resolving forces horizontally along $\rightarrow$ we have

$$P - 2P\cos\frac{\pi}{6} + L\cos\frac{\pi}{3} + 3P - 4P\cos\frac{\pi}{6} + M\cos\frac{\pi}{3} + N\cos\frac{\pi}{3} + 5P = 0 \quad (10)$$

$$9P - 3\sqrt{3}P + \frac{1}{2}L + \frac{1}{2}M + \frac{1}{2}N = 0$$

i.e. $L + M + N = (\sqrt{3}-3)6P \rightarrow (1)$ (05)

Resolving forces vertically upwards we have




$$-2P\cos\frac{\pi}{3} - 4P\cos\frac{\pi}{3} - M\sin\frac{\pi}{3} + N\sin\frac{\pi}{3} + L\sin\frac{\pi}{3} = 0 \quad (10)$$

$$-3P - \frac{\sqrt{3}}{2}(M - N - L) = 0$$

i.e. $L - M + N = 2\sqrt{3}P \rightarrow (2)$ (05)

Taking moment about $F$ in the anti-clockwise we have

$$-2P.2a + P.2a\sin\frac{\pi}{3} - 5P.2a\sin\frac{\pi}{3} + L.2.2a\sin\frac{\pi}{3} = 0 \quad (10)$$

$$-2P - 2\sqrt{3}P + \sqrt{3}L = 0$$



$$L = \frac{2}{\sqrt{3}}\left(1+\sqrt{3}\right)P \quad (10)$$

$(1)-(2) \Rightarrow \quad 2M = (\sqrt{3}-3)6P - 2\sqrt{3}P = \left(4\sqrt{3}-18\right)P \Rightarrow M = \left(2\sqrt{3}-9\right)P$ (10)

$(1)+(2) \Rightarrow$

$$2N = (\sqrt{3}-3)6P + 2\sqrt{3}P - 2L = \left(8\sqrt{3}-18\right)P - \frac{4}{\sqrt{3}}\left(1+\sqrt{3}\right)P = \left(\frac{20\sqrt{3}}{3} - 22\right)P$$

$$N = \left(\frac{10}{\sqrt{3}} - 11\right)P \quad (10)$$

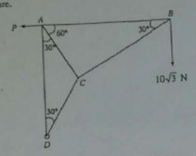
[70]

15. (a) Two uniform rods $AB$ and $BC$ are equal in length. The weight of $AB$ is $2w$ and the weight of $BC$ is $w$. The rods are smoothly hinged at $B$ and the midpoints of the rods are connected by a light inelastic string. The system stands in equilibrium in a vertical plane with $A$ and $C$ on a smooth horizontal table.

If $A\hat{B}C = 2\theta$, show that the tension of the string is $\frac{3}{2}w\tan\theta$.

Find the magnitude of the reaction at $B$ and the angle it makes with the horizontal.

(b) Five light rods $AB$, $BC$, $CD$, $DA$ and $AC$ are smoothly jointed at their ends to form a framework as shown in the figure.




$A\hat{B}C = A\hat{D}C = D\hat{A}C = 30^\circ$ and $B\hat{A}C = 60^\circ$. The framework is smoothly hinged at $D$ and carries a weight of $10\sqrt{3}$ newtons at $B$. The framework is held in a vertical plane, with $AB$ horizontal, by a horizontal force of $P$ newtons at $A$.

(i) Find the value of $P$.

(ii) Find the magnitude and the direction of the reaction at $D$.

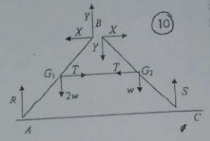
(iii) Using Bow's notation, draw a stress diagram for the framework and find the stresses in all the rods, distinguishing between tensions and thrusts.

(a) Let $AB = BC = 2a$

Taking moment about $C$ in the anticlockwise sense for the system we have

$w.a.\sin\theta + 2w.3a.\sin\theta - R.4a.\sin\theta = 0$ (15)

$R = \frac{7}{4}w$. (05)




Taking moment about $B$ in the anticlockwise sense for the rod $AB$ we have

$T.a.\cos\theta + 2w.a.\sin\theta - R.2a.\sin\theta = 0$ (15)

$T = -2w\tan\theta + 2R.\tan\theta = \left(-2w + \frac{7}{2}w\right)\tan\theta = \frac{3}{2}w\tan\theta$ (05) [40] $\boxed{50}$

Resolving horizontally for the rod $AB$ we have

$X = T = \frac{3}{2}w\tan\theta$ (05)

Resolving vertically for the rod $AB$ we have

$Y + R - 2w = 0$ (05)

$Y = -R + 2w = -\frac{7}{4}w + 2w = \frac{1}{4}w$. (05)

Thus, the reaction at the join $B$

$\sqrt{X^2 + Y^2} = \sqrt{\left(\frac{3}{2}w\tan\theta\right)^2 + \left(\frac{1}{4}w\right)^2} = \frac{w}{4}\sqrt{1 + 36\tan^2\theta}$ (05) [20] $\boxed{15}$



The angle that the reaction makes with the horizontal is

$\tan^{-1}\left(\frac{Y}{X}\right) = \tan^{-1}\left(\frac{\frac{w}{4}}{\frac{3}{2}w\tan\theta}\right) = \tan^{-1}\left(\frac{1}{6}\cot\theta\right)$ (10) $\boxed{5}$ [15]

(b) (i) Taking moments about $D$ we have

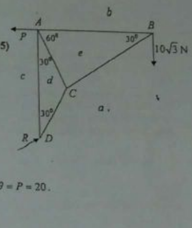
$P.AD - 10\sqrt{3}.AB = 0$ (05)

But $AD = 2AC\cos 30^\circ$

$= 2AB\cos 60^\circ \cos 30^\circ = \frac{\sqrt{3}}{2}AB$ (05)

$P.\frac{\sqrt{3}}{2}AB - 10\sqrt{3}.AB = 0$

$P = 20\text{N}$. (05) [15]




Let $R$ be the reaction at $E$ and $\theta$ be the angle that $R$ makes with the horizontal.

Resolving forces vertically we have

$R\sin\theta = 10\sqrt{3}$. (05)

Resolving forces horizontally we have $R\cos\theta = P = 20$.

$R = \sqrt{(10\sqrt{3})^2 + 20^2} = 10\sqrt{7}\text{ N}$. (05)

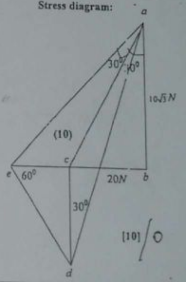
$\tan\theta = \frac{10\sqrt{3}}{20} = \frac{\sqrt{3}}{2}$ (05)

$\theta = \tan^{-1}\frac{\sqrt{3}}{2}$

Since the system is in equilibrium under three forces the reaction $R$ should also passes through $B$.

[15]

**Stress diagram:**




[10] / 0

| Rod | Stress | Magnitude |
|---|---|---|
| $AB$ | Tension | 30 N |
| $BC$ | Thrust | $20\sqrt{3}$ N |
| $AC$ | Thrust | 20 N |
| $DC$ | Thrust | 40 N |
| $AD$ | Tension | $10\sqrt{3}$ N |

(20) (20) [40]

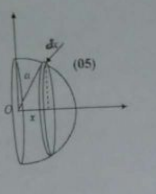
✓4 → 15
✓3 → 10
✓2 → 05

16. Show that the centre of mass of a uniform solid hemisphere of radius $a$ is on its axis of symmetry at a distance $\frac{3}{8}a$ from the base of the hemisphere.

The inner and outer radii of a uniform solid hemispherical shell are $a$ and $b$ $(> a)$. Show that the distance of its centre of mass from the centre along the axis of symmetry is $\frac{3(a+b)(a^2+b^2)}{8(a^2+ab+b^2)}$

This hemispherical shell rests in equilibrium so that its curved surface is in contact with a rough horizontal ground and equally rough vertical wall.

Show that if the equilibrium is limiting, the inclination of the base to the horizontal is $\sin^{-1}\left\{\frac{8\mu b(1+\mu)(a^2+ab+b^2)}{3(1+\mu^2)(a+b)(a^2+b^2)}\right\}$, where $\mu$ is the coefficient of friction between the shell and the rough surfaces.



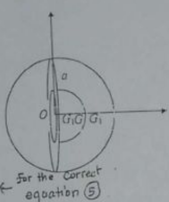

By symmetry the centre of mass of the hemisphere lies on the axis of symmetry.
(05)
Let $\bar{x}$ be the distance to the centre of mass of the hemisphere from $O$ the centre of the base of the hemisphere.
Let $\rho$ be the density of the hemisphere.

$\frac{1}{2}\cdot\frac{4}{3}\pi a^3 \rho \bar{x}$ (10)

$= \int_0^a \pi(a^2-x^2)x\rho\, dx$ (10)

i.e. $\frac{2}{3}a^3\bar{x} = \int_0^a (a^2x - x^3)dx = \left[a^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^a = \left(\frac{1}{2}-\frac{1}{4}\right)a^4 = \frac{1}{4}a^4 \Rightarrow \bar{x} = \frac{3}{8}a$. (05)
(05) (05)

Thus, the centre of mass of a uniform solid hemisphere of radius $a$ is at a distance $\frac{3}{8}a$ from the base of the hemisphere. [45]



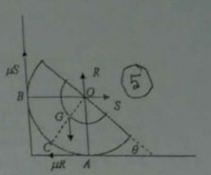

The centre of mass of the solid hemisphere of radius $a$ is at a distance $\frac{3}{8}a$ from $O$. (05)
The centre of mass of the solid hemisphere of radius $b$ is at a distance $\frac{3}{8}b$ from $O$. (05)

Let $\bar{\bar{x}}$ be the distance to the centre of mass of the shell from $O$.

$\left(\frac{2}{3}\pi b^3 - \frac{2}{3}\pi a^3\right)\rho\bar{\bar{x}} = \left(\frac{2}{3}\pi b^3\right)\rho\frac{3}{8}b - \left(\frac{2}{3}\pi a^3\right)\rho\frac{3}{8}a$. ← For the correct equation 5
(10) (10) (10)

i.e. $\bar{\bar{x}} = \frac{\frac{3}{8}(b^4-a^4)}{(b^3-a^3)} = \frac{3}{8}\cdot\frac{(a+b)(a^2+b^2)}{a^2+ab+b^2} \rightarrow$ (1). (05)
05
[45]

Resolving horizontally we have $S = \mu R \rightarrow$ (1) (05)
Resolving vertically we have $R + \mu S = w \rightarrow$ (2) (05)
From (1) and (2) we have
$R = \frac{w}{1+\mu^2}$ and $S = \frac{\mu w}{1+\mu^2}$
(05) (05)
Taking moment about $O$ we have
$w OG \sin\theta = \mu R.OA + \mu S.OB$ (15)




i.e. $w\frac{3}{8}\frac{(a+b)(a^2+b^2)}{a^2+ab+b^2}\sin\theta = \frac{\mu w}{1+\mu^2}b + \frac{\mu^2 w}{1+\mu^2}b$ (10)

i.e. $\frac{3}{8}\frac{(a+b)(a^2+b^2)}{a^2+ab+b^2}\sin\theta = \frac{\mu(1+\mu)}{1+\mu^2}b$ (05)

i.e. $\sin\theta = \frac{8}{3}\frac{\mu b(1+\mu)(a^2+ab+b^2)}{(1+\mu^2)(a+b)(a^2+b^2)}$ (05)

$\theta = \sin^{-1}\left\{\frac{8}{3}\cdot\frac{\mu b(1+\mu)(a^2+ab+b^2)}{(1+\mu^2)(a+b)(a^2+b^2)}\right\}$ $\cancel{(05)}$ [60]

17. (a) Nimal, Sunil and Piyal play a game with a biased coin which has probability $p$ of landing a head. Nimal, Sunil and Piyal in that order toss the coin in turns. The first person who gets a tail will win the game.

Find the probability that Nimal wins the game in his

(i) second turn,

(ii) third turn.

**Hence**, find the probability that Nimal wins the game eventually.

**Deduce** that, if the coin is more likely to land tails than heads, Nimal has more than 50% chance of winning the game.

(b) The mean and the standard deviation of a set of observations $\{x_1, x_2, \ldots, x_n\}$ are $\bar{x}$ and $s_x$ respectively. Suppose that a linear transformation $y_i = a + bx_i$, where $a$ and $b$ are constants, transforms the original data set $\{x_1, x_2, \ldots, x_n\}$ to the set $\{y_1, y_2, \ldots, y_n\}$.

Show that $\bar{y} = a + b\bar{x}$ and $s_y^2 = b^2 s_x^2$, where $\bar{y}$ and $s_y$ are the mean and the standard deviation of the set $\{y_1, y_2, \ldots, y_n\}$.

(i) Find the mean and the standard deviation of the set of observations $\{1, 2, 3, 4, 5, 6, 7\}$.

Hence, find

(α) the mean and the standard deviation of the set of observations $\{2.01, 3.02, 4.03, 5.04, 6.05, 7.06, 8.07\}$.

(β) seven values whose mean is 5 and the standard deviation is 6.

(ii) Salt is packed in bags which the manufacturer claims contain 25 kg each. The following information is given for 80 such bags whose actual weights are not known:

$\sum_{i=1}^{80}(x_i - 25) = 27.2$ and $\sum_{i=1}^{80}(x_i - 25)^2 = 85.1$, where $x_i\,(i = 1, 2, \ldots, 80)$ denotes the actual weight of the $i^{th}$ bag. Using an appropriate linear transformation or otherwise, find the mean and the variance of the actual weights of the eighty bags.

(a) Let $N$ – for Nimal getting a tail, $S$ – for Sunil getting a tail and $P$ – for Piyal getting a tail.

Then $P(N) = P(S) = P(P) = 1 - p = q$ (say). (05)

$P(N'S'P'N)$ 5    $PPP(1-p) = p^3(1-p)$ 05    5    $\boxed{20}$

(iii) $P$(Nimal wins the game eventually)

$= P(N \cup N'S'P'N \cup \cancel{N'S'P'N'S'P'N} \cup \cdots)$ (05)

$= P(N) + P(N'S'P'N) + P(N'S'P'N'S'P'N) + \cdots$ (05)

$= P(N) + P(N')P(S')P(P')P(N) + P(N')P(S')P(P')P(N')P(S')P(P')P(N) + \cdots$ (05)

$P(N'S'P'N'S'P'N)$ 05

$= q + p^3q + p^6q + \cdots$ (05)

05 [30] $= p^6(1-p)$ 05    $\boxed{15}$

$= q(1 + p^3 + p^6 + \cdots) = q\dfrac{1}{1-p^3} = \dfrac{1}{1+p+p^2}$. (05) 05    $\boxed{25}$

(ii) $P$(Piyal wins the game)

$= P(N'S'P \cup N'S'P'N'S'P \cup N'S'P'N'S'P'N'S'P \cup \cdots)$ (05)

$= P(N'S'P) + P(N'S'P'N'S'P) + P(N'S'P'N'S'P'N'S'P) + \cdots$ (05)

$= P(N')P(S')P(P) + P(N')P(S')P(P')P(N')P(S')P(P)$
$\quad + P(N')P(S')P(P')P(N')P(S')P(P')P(N')P(S')P(P) + \cdots$

$= p^2q + p^5q + p^8q + \cdots$ (05)

$= p^2q(1 + p^3 + p^6 + \cdots) = qp^2\dfrac{1}{1-p^3} = \dfrac{p^2}{1+p+p^2}$.    ~~[25]~~

(05)    (05)

If the coin is such that it is more likely to lands tails than heads, then $p < \dfrac{1}{2}$.

Thus, we have $1 + p + p^2 < 1 + \dfrac{1}{2} + \dfrac{1}{4} = \dfrac{7}{4}$. (05)



$\therefore P(\text{Nimal wins the game}) = \dfrac{1}{1+p+p^2} > \dfrac{4}{7} = 0.571 > 0.5$

(05)    (05)

Nimal has more than 50% chance of wining the game. [15]

(b) $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n}(a + bx_i) = na + b\sum_{i=1}^{n} x_i$. (05)

$\dfrac{1}{n}\sum_{i=1}^{n} y_i = a + \dfrac{b}{n}\sum_{i=1}^{n} x_i$

i.e., $\bar{y} = a + b\bar{x}$ (05) [10]

$S_y^2 = \dfrac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2 = \dfrac{1}{n}\sum_{i=1}^{n}\{(a + bx_i) - (a + b\bar{x})\}^2 = \dfrac{b^2}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = b^2 S_x^2$

(05)    (05)    [10]

$\bar{x} = \dfrac{1+2+3+4+5+6+7}{7} = \dfrac{28}{7} = 4$. (05) [05]

$s_x = \sqrt{\dfrac{1}{7}\{(1-4)^2 + (2-4)^2 + (3-4)^2 + (4-4)^2 + (5-4)^2 + (6-4)^2 + (7-4)^2\}}$

$= \sqrt{\dfrac{1}{7}\{9+4+1+1+4+9\}} = \sqrt{\dfrac{28}{7}} = 2$. (05) [05]

(i) $y = 1 + 1.01x$ will transforms the set $\{1, 2, 3, 4, 5, 6, 7\}$ into the set $\{2.01, 3.02, 4.03, 5.04, 6.05, 7.06, 8.07\}$. (05)

$\bar{y} = 1 + 1.01 \times 4 = 5.04$ (05)

$s_y = bs_x = 1.01 \times 2 = 2.02$ (05) [15]

5    $\boxed{20}$

(ii) Consider the transformation $y = a + bx$, where $a$ and $b$ are constants.

Then we have

$\bar{y} = a + b\bar{x}$ and $s_y = bs_x$

Thus we get

$5 = a + 4b$ and $6 = 2b \Rightarrow b = 3$ and $a = -7$.

(05)    (05)

$y = -7 + 3x$

The seven numbers are -4, -1, 2, 5, 8, 11, 14. (05) [15]

05    05    $\boxed{15}$

(c) Consider the transformation $y = -25 + x$. (05)

Then from $\bar{y} = -25 + \bar{x}$ we have $\dfrac{27.2}{80} = -25 + \bar{x} \Rightarrow \bar{x} = 25 + 0.34 = 25.34$

(05)    (05)    $\boxed{15}$ ~~[10]~~

From $s_y^2 = s_x^2$ we have $s_x = \sqrt{\dfrac{85.1}{80}} = \sqrt{1.06375} = 1.031$

~~(05)~~    (05)    ~~[10]~~ $\boxed{05}$