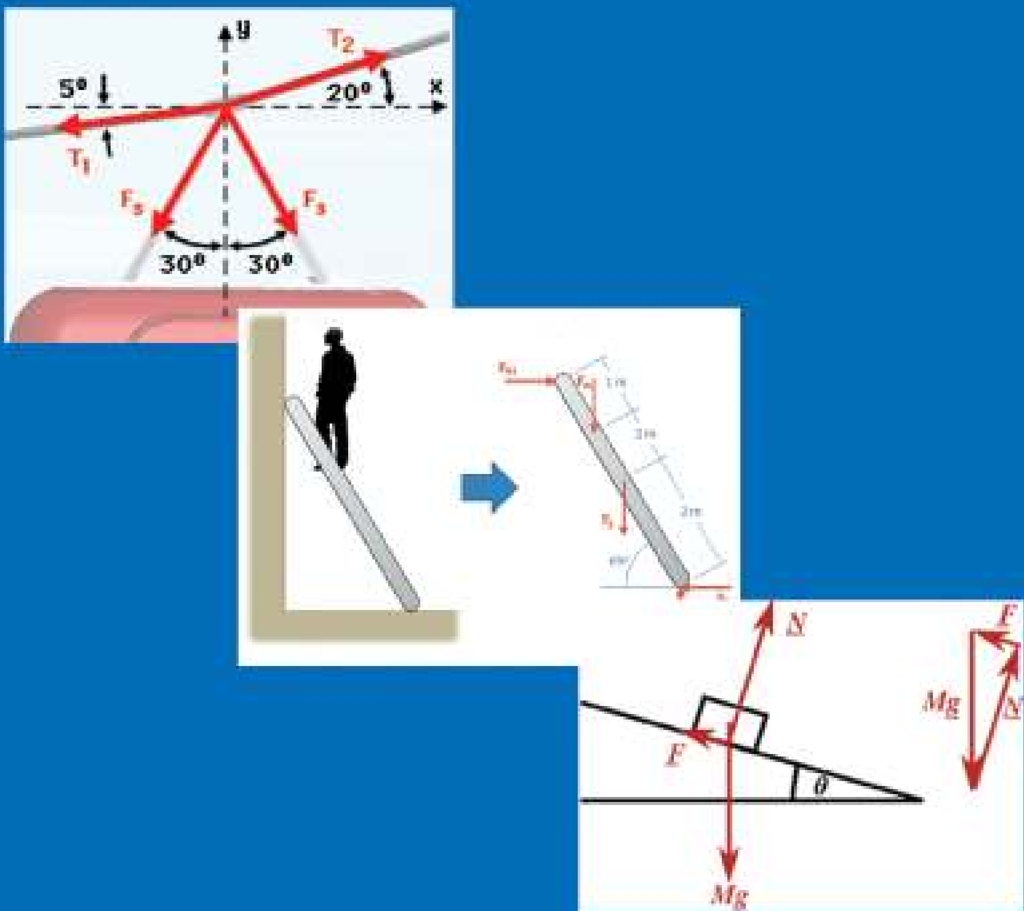
G.C.E. Advanced Level

Combined Mathematics

STATICS – I

Additional Reading Book

(Prepared According to the New syllabus Implemented From 2017)



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National Institute of Education
Maharagama
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STATICS - I

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Department of Mathematics
Faculty of Science and Technology
National Institute of Education
Maharagama

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Message from the Director General

Department of Mathematics of National Institute of Education time to time implements many different activities to develop the mathematics education. The publication of this book is a mile stone which was written in the name of **“Statics - Part I, Statics - Part II”.**

After learning of grade 12 and 13 syllabus, teachers should have prepared the students for the General Certificate of Education (Advanced Level) which is the main purpose of them. It has not enough appropriate teaching - learning tools for the proper utilization. It is well known to all, most of the instruments available in the market are not appropriate for the use and it has not enough quality in the questions. Therefore **“Statics - Part I, Statics - Part II”.** book was prepared by the Department of Mathematics of National Institute of Education which was to change of the situation and to ameliorate the students for the examination. According to the syllabus the book is prepared for the reference and valuable book for reading. Worked examples are included which will be helpful to the teachers and the students.

I kindly request the teachers and the students to utilize this book for the mathematics subjects' to enhance the teaching and learning process effectively. My gratitude goes to Aus Aid project for sponsoring and immense contribution of the internal and external resource persons from the Department of Mathemetics for toil hard for the book of **“Statics - Part I, Statics - Part II”.**

Dr. (Mrs). T. A. R. J. Gunasekara
Director General
National Institute of Education.

Message from the Director

Mathematics holds a special place among the G.C.E. (A/L) public examination prefer to the mathematical subject area. The footprints of the past history record that the country's as well as the world's inventor's spring from the mathematical stream.

The aim and objectives of designing the syllabus for the mathematics stream is to prepare the students to become experts in the Mathematical, Scientific and Technological world.

From 2017 the Combined Mathematics syllabus has been revised and implemented. To make the teaching - learning of these subjects easy, the Department of Mathemactics of National Institute of Education has prepared **Statics - Part 1 and Part 11** as the supplementary reading books. There is no doubt that the exercises in these books will measure their achievement level and will help the students to prepare themselves for the examination. By practicing the questions in these books the students will get the experience of the methods of answering the questions. Through the practice of these questions, the students will develop their talent, ability, skills and knowledge. The teachers who are experts in the subject matter and the scholars who design the syllabus, pooled their resources to prepare these supplementary reading books. While preparing these books, much care has been taken that the students will be guided to focus their attention from different angles and develop their knowledge. Besides, the books will help the students for self-learning.

I sincerely thank the Director General for the guidance and support extented and the resource personnel for the immense contribution. I will deeply appreciate any feedback that will shape the reprint of the books.

Mr. K. R. Pathmasiri
Director
Department of Mathematics

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Preface

This book is being prepared for the students of Combined Mathematics G.C.E.A/L to get familiar with the subject area of Statics. It is a supplementary book meant for the students to get practice in answering the questions for self learning. The teachers and the students are kindly invited to understand, it is not a bunch of model questions but a supplementary to encourage the students towards self learning and to help the students who have missed any area in the subject matter to rectify them.

The students are called upon to pay attention that after answering the questions in worked examples by themselves, they can compare their answers with the answers given in the book. But it is not necessary that all the steps have taken to arrive at the answers should tally with the steps mentioned in the book's answers given in this book are only a guide.

Statics - Part I is released in support of the revised syllabus - 2017. The book targets the students who will sit for the GCE A/L examination – 2019 onwards. The Department of Mathematics of National Institute of Education already released **Practice Questions and Answers** book and it is being proceeded by the **" Statics I".** There are other books soon be released with the questions taken Unit wise **"Questions bank"** and **" Statics - Part II".**

We shall deeply appreciate your feedback that will contribute to the reprint of this book.

Mr.S.Rajendram
Project Leader
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Content

1.0 Vectors

1.1 Scalar quantities

Quantities which can be entirely determined by numbers with appropriate units are called scalar quantities.

Distance, time, mass, volume, temperature are scalar quantities.

Further, two quantities of the same kind, when added will give another quantity of the same kind.

Examples:

Mass is 10 kg; Temperature is 27° C, Time is 20 s. Length is 2 m; Area is 5 m^2, Volume is 4 m^3, capacity is 2 *l*, speed is 5 m s^{-1}. The numerical parts of the above examples without the units are called **Scalars.**

There are also quantities which cannot be described fully by magnitude (with units) alone but which can be known fully by magnitute and direction. For example,

i) A ship is travelling with a speed of 15 km h^{-1} due north.

ii) A force 20 newton act on a particle vertically downwards.

Study on vectors was first focused in middle of 19th century. In the recent past "Vectors" has become an indispensable tool, used in the mathematical calculation of engineers, mathematician and physicists while physical and geometrical problems can be expressed concisely, by using vectors.

1.2 Vector quantities

Quantities which can be described completly by magnitudes (with units) and directions are called vector quantitics.

Examples:

i. Displacement due north is 5 m.

ii. Velocity is 15 m s^{-1} due south east.

iii. Weight is 30 N, Vertically downwards.

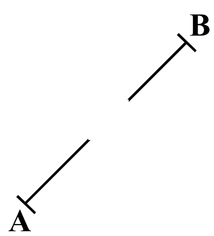
iv. Force of 10 N inclined upwards 30° to the horizontal.

Vectors have both magnitude and direction.

1.3 Representation of vectors

There are two ways of representing vectors.

Geometrical Representation

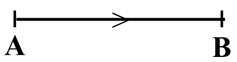


A vector can be represented by a **directed line segment** $\overrightarrow{AB}$

The length of the line segment will give the magnitude of the vector and the arrow head on it denotes the direction. This is said to be the geometrical representation of a vector.

Example:

To denote a force of 4 N due east a straight line segment AB is drawn towards east where AB = 4 units. The direction of the force is denoted by the arrow from A to B as shown below.



Algebraic Representation

The vector $\overrightarrow{AB}$ is denoted by a single algebraic symbol such as $\underline{\boldsymbol{a}}$ or $\overline{\boldsymbol{a}}$. In some text books generally it is denoted by the symbol **a** in dark print.

1.4 Modulus of a vector

The magnitude of a vector is known as its modulus.

The modulus of a vector $\overrightarrow{AB}$ (or $\underline{\boldsymbol{a}}$) is denote by $\left|\overrightarrow{AB}\right|$ or $|\underline{\boldsymbol{a}}|$

The modulus of a vector is always non- negative.

1.5 Equality of two vectors

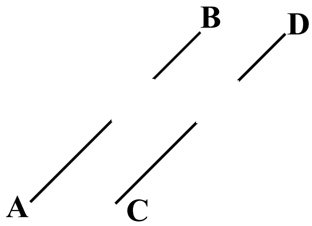
If two vectors are equal in magnitude and are in the same direction they are called equal vectors.

The two vectors $\overrightarrow{AB}$ (= $\underline{\boldsymbol{a}}$) and $\overrightarrow{CD}$ (=$\underline{\boldsymbol{b}}$) are equal if and only if

i) $\left|\overrightarrow{AB}\right| = \left|\overrightarrow{CD}\right|$

ii) $AB \ // \ CD$ and

(iii) $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are in the same direction.



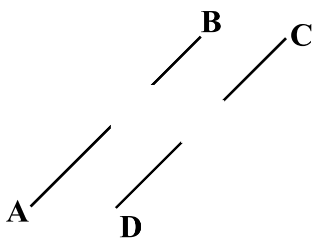
Note : Consider the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$

AB = CD i.e $\left|\overrightarrow{AB}\right| = \left|\overrightarrow{CD}\right|$

$AB \ // \ CD$

But they are not in the same direction.

Therefore $\overrightarrow{AB} \neq \overrightarrow{CD}$; $\underline{\boldsymbol{a}} \neq \underline{\boldsymbol{b}}$



1.6 Unit vector

A vector with unit magnitude is called unit vector. Given a vector $\underline{\boldsymbol{a}}$, the unit vector in the direction of $\boldsymbol{a}$ is $\frac{\underline{\boldsymbol{a}}}{|\underline{\boldsymbol{a}}|}$ is denoted by $\hat{\underline{\boldsymbol{a}}}$.

1.7 Zero vector (null vector)

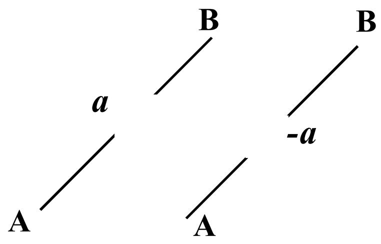
A vector with zero magnitude is called zero vector. It is denoted by **0**. $|\mathbf{0}| = 0$ and its direction is arbitrary and is represented by a point.

1.8 Negative vector of a given vector

Given a vector $\overrightarrow{AB}$, the vector $\overrightarrow{BA}$ is negative vector of $\overrightarrow{AB}$ and is written $\overrightarrow{BA} = -\overrightarrow{AB}$.

If $\overrightarrow{AB} = \underline{\boldsymbol{a}}$, then $\overrightarrow{BA} = -\underline{\boldsymbol{a}}$

$|\overrightarrow{AB}| = |\overrightarrow{BA}|$, $|\underline{a}| = |-\underline{a}|$



1.9 Scalar multiple of a vector

When $\underline{\boldsymbol{a}}$ is a vector and λ is a scalar, then $\lambda\underline{\boldsymbol{a}}$ is the product of the vector $\underline{\boldsymbol{a}}$ and scalar λ. Here λ should be considerd under three cases namely when $\lambda > \mathbf{0}$, $\lambda = \mathbf{0}$ and $\lambda < \mathbf{0}$.

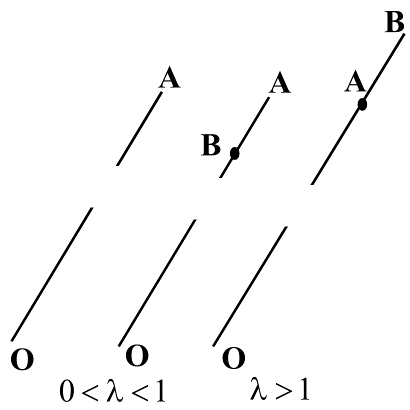
Case (i) $\lambda > 0$

Let $\overrightarrow{OA} = \underline{\boldsymbol{a}}$,

Take a point B on OA.

(or produced OA) such that $OB = \lambda OA$

$\overrightarrow{OB} = \lambda\overrightarrow{OA} = \lambda\underline{\mathbf{a}}$

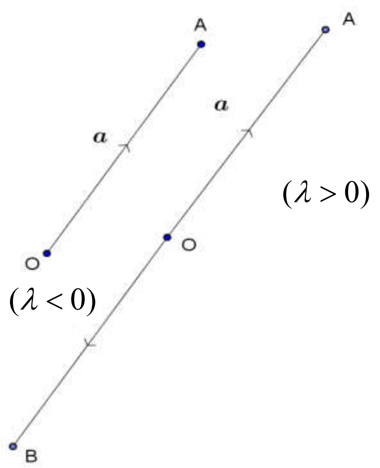


(ii) When $\lambda = \mathbf{0}$, $\lambda\underline{\mathbf{a}}$ is defind as the nullvector.

That is $\lambda\underline{\mathbf{a}} = 0\underline{\mathbf{a}} = \boldsymbol{0}$

(iii) $\lambda < 0$

In this case$\lambda\underline{\mathbf{a}}$ is a vector opposite to the direction of **a** with a magnitude of $|\lambda|$ times OA. Choose a point B on AO produced such that $OB = |\lambda|\, OA$. Then $\overrightarrow{OB} = \lambda\underline{\mathbf{a}}$.

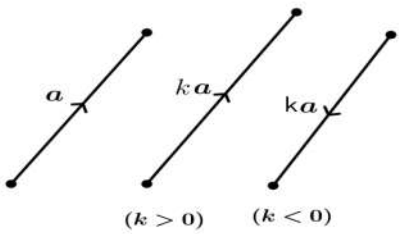


1.10 Parallel vectors

Given a vector $\underline{a}$ and $k\underline{a}$, $k\underline{a}$ is a vector parallel to $\underline{a}$

(i) When $k > \mathbf{0}$, the vector $k\underline{a}$ is in the direction of $\underline{a}$

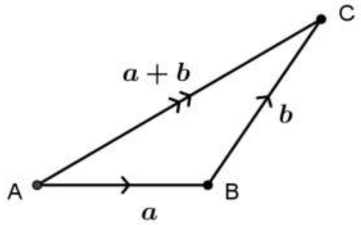
(ii) When $k < \mathbf{0}$, the vector $k\underline{a}$ is opposite in direction to that of $\underline{a}$.



Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are said to be parallel. if $\underline{b} = \lambda \underline{a}$

1.11 Vector addition

If two vectors $\underline{a}$ and $\underline{b}$ are represented by $\overrightarrow{AB}$ and $\overrightarrow{BC}$ respectively then the vector addtion of $\underline{a}$ and $\underline{b}$ is represented by $\overrightarrow{AC}$

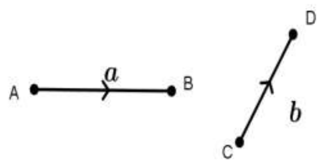


ie. $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$= \underline{a} + \underline{b}$

This is called the triangle law of vector addition.

Let $\overrightarrow{AB} = \underline{a}$ and $\overrightarrow{CD} = \underline{b}$ be two vectors.



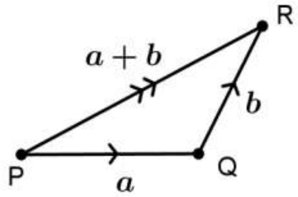
Draw a line segment PQ such that PQ = AB and PQ // AB.

Draw a line segment QR such that QR = CD and QR // CD.

By definition $\overrightarrow{AB} = \overrightarrow{PQ} = \underline{a}$ and $\overrightarrow{CD} = \overrightarrow{QR} = \underline{b}$

According to the triangle of law of addition

$\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR} = \underline{a} + \underline{b}$



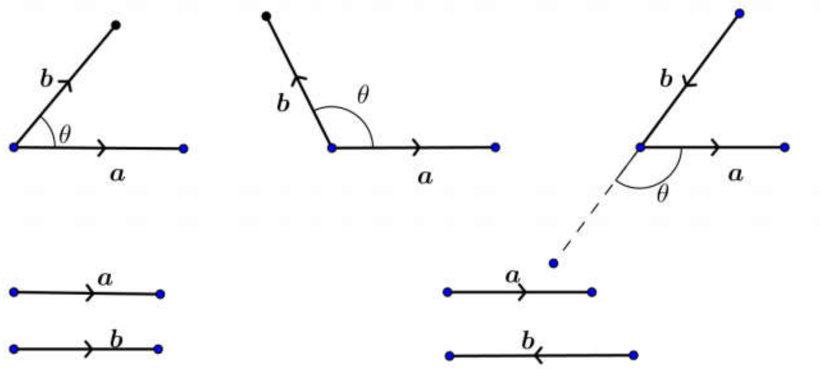
1.12 Definition of a vector

A vector has magnitude and direction and obeys the triangle of law of addition.

1.13 Angle between two vectors

Let $\underline{a}$ and $\underline{b}$ be two vectors.

The angle θ between $\underline{a}$ and $\underline{b}$ is shown below.



Note that $0 \le \theta \le \pi$

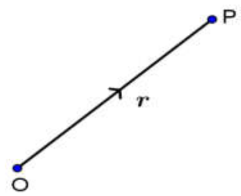
If $\underline{a}$ and $\underline{b}$ are parallel and are in the same direction, then $\theta = 0$.

If $\underline{a}$ and $\underline{b}$ are parallel and are in the opposite direction, then $\theta = \pi$.

1.14 Position vector

With a fixed point O chosen as the origin, the position of any point P can be denoted by the vector $\overrightarrow{OP}$.

The vector $\overrightarrow{OP} = \underline{r}$ is (known as the) position vector of P with respect to O.

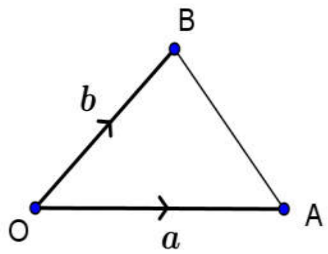


Let the position vectors of two points A and B be $\underline{a}$ and $\underline{b}$.

$\overrightarrow{OA} = \underline{a}$, $\overrightarrow{OB} = \underline{b}$

$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= \underline{b} - \underline{a}\end{aligned}$$



1.15 Laws of vector algebra

Let $\underline{a}$, $\underline{b}$, $\underline{c}$ be vectors and λ, μ be scalars.

(i) $\underline{a} + \underline{b} = \underline{b} + \underline{a}$ (Commutative Law)

(ii) $(\underline{a} + \underline{b}) + \underline{c} = \underline{a} + (\underline{b} + \underline{c})$ (Associative Law)

(iii) $\lambda(\underline{a} + \underline{b}) = \lambda\underline{a} + \lambda\underline{b}$ (Distributive Law)

(iv) $\underline{a} + \underline{0} = \underline{a} = \underline{0} + \underline{a}$

(v) $\underline{a} + (-\underline{a}) = \underline{0} = (-\underline{a}) + \underline{a}$

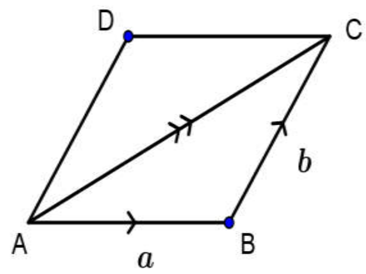
(vi) $(\lambda + \mu)\underline{a} = \lambda\underline{a} + \mu\underline{a}$

(vii) $\lambda\mu(\underline{a}) = \lambda[\mu\underline{a}] = \mu[\lambda\underline{a}]$

proof :

(i) Let $\overrightarrow{AB} = \underline{\boldsymbol{a}}$ and $\overrightarrow{BC} = \underline{\boldsymbol{b}}$

Complete the parallelogram ABCD



Now $\overrightarrow{DC} = \overrightarrow{AB} = \underline{\boldsymbol{a}}$

$\overrightarrow{AD} = \overrightarrow{BC} = \underline{\mathbf{b}}$

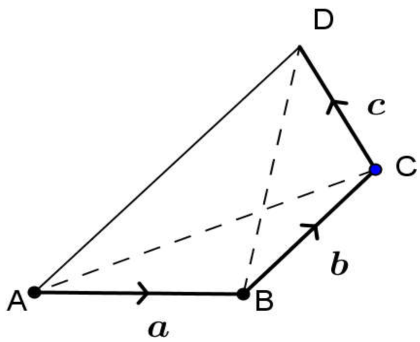
By triangle law of vector addition

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}$

$\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \underline{\boldsymbol{b}} + \underline{\boldsymbol{a}}$

Hence $\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}} = \underline{\boldsymbol{b}} + \underline{\boldsymbol{a}}$

(ii) Let $\overrightarrow{AB} = \underline{\boldsymbol{a}}$, $\overrightarrow{BC} = \underline{\boldsymbol{b}}$ and $\overrightarrow{CD} = \underline{\boldsymbol{c}}$



$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD}$

$= \overrightarrow{AB} + (\overrightarrow{BC} + \overrightarrow{CD})$

$= \underline{\boldsymbol{a}} + (\underline{\boldsymbol{b}} + \underline{\boldsymbol{c}})$ .... ①

$\overrightarrow{AD} = \overrightarrow{AC} + \overrightarrow{CD}$

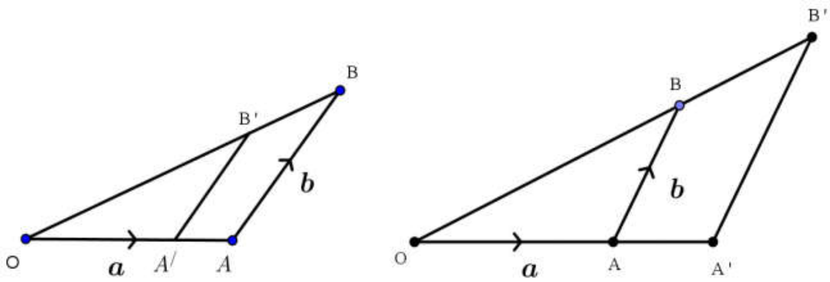
$= (\overrightarrow{AB} + \overrightarrow{BC}) + \overrightarrow{CD}$

$= (\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}) + \underline{\boldsymbol{c}}$ ....②

From (1) and (2) $(\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}) + \underline{\boldsymbol{c}} = \underline{\boldsymbol{a}} + (\underline{\boldsymbol{b}} + \underline{\boldsymbol{c}})$

(iii) $\lambda(\underline{\boldsymbol{a}} + \underline{\mathbf{b}}) = \lambda\underline{\boldsymbol{a}} + \lambda\underline{\mathbf{b}}$

Let $\overrightarrow{OA} = \underline{\boldsymbol{a}}$ and $\overrightarrow{AB} = \underline{\boldsymbol{b}}$



$0 < \lambda < 1$ $\qquad$ $\lambda > 1$

Take the point A' on OA (or OA produced)

such that $\overrightarrow{OA'} = \lambda\,\overrightarrow{OA} = \lambda\underline{\mathbf{a}}$

The line drawn parallel to AB through A' meet OB (or OB produed) at B'.

Now, ΔOAB, $\Delta OA'B'$ are similar triangles.

$$\frac{OA'}{OA} = \frac{A'B'}{AB} = \frac{OB'}{OB} = \lambda.$$

$\overrightarrow{A'B'} = \lambda\,\overrightarrow{AB} = \lambda\underline{\boldsymbol{b}}$ and $\overrightarrow{OB'} = \lambda\,\overrightarrow{OB}$ .... ①

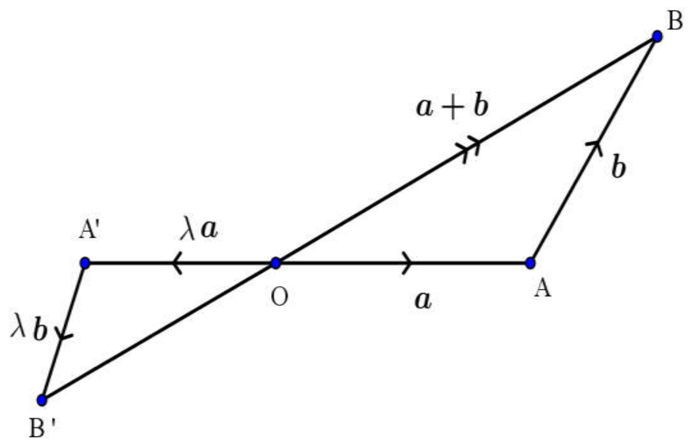
$\overrightarrow{OB'} = \overrightarrow{OA'} + \overrightarrow{A'B'} = \lambda\underline{\boldsymbol{a}} + \lambda\underline{\mathbf{b}}$ ....②

$\lambda\,\overrightarrow{OB} = \lambda(\overrightarrow{OA} + \overrightarrow{AB}) = \lambda(\underline{\boldsymbol{a}} + \underline{\mathbf{b}})$ ....③

From (i) $\overrightarrow{OB'} = \lambda\,\overrightarrow{OB}$

$\lambda\underline{\boldsymbol{a}} + \lambda\underline{\mathbf{b}} = \lambda(\underline{\boldsymbol{a}} + \underline{\mathbf{b}})$

When $\lambda < 0$



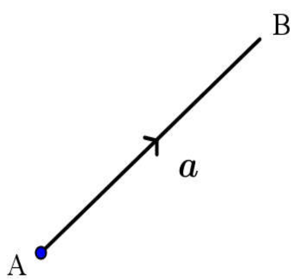
$\overrightarrow{OA} = \underline{\boldsymbol{a}}$, $\overrightarrow{AB} = \underline{\boldsymbol{b}}$, $\overrightarrow{OA'} = \lambda\underline{a}$

A'B' is drawn parallel to BA and meets BO produced at B'. $\overrightarrow{A'B'} = \lambda\underline{\mathbf{b}}$ and $\overrightarrow{OB'} = \lambda\underline{\boldsymbol{a}} + \lambda\underline{\mathbf{b}}$

By the properties of similar triangles and vectors it can be easily proved that

$\lambda(\underline{\boldsymbol{a}} + \underline{\mathbf{b}}) = \lambda\underline{\boldsymbol{a}} + \lambda\underline{\mathbf{b}}$ when $\lambda < 0$

(iv) $\underline{\boldsymbol{a}} + \underline{\boldsymbol{0}} = \underline{\boldsymbol{a}} = \underline{\boldsymbol{0}} + \boldsymbol{a}$



Let $\overrightarrow{AB} = \underline{\boldsymbol{a}}$

$\overrightarrow{AB} = \overrightarrow{AB} + \overrightarrow{BB}$

$\underline{\boldsymbol{a}} = \underline{\boldsymbol{a}} + \underline{\boldsymbol{0}}$ .... ①

$\overrightarrow{AB} = \overrightarrow{AA} + \overrightarrow{AB}$

$\underline{\boldsymbol{a}} = \underline{\boldsymbol{0}} + \underline{\boldsymbol{a}}$ .... ②

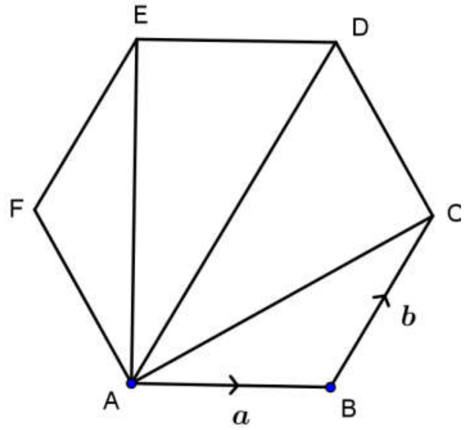
From (1) and (2) $\underline{\boldsymbol{a}} + \underline{\boldsymbol{0}} = \underline{\boldsymbol{a}} = \underline{\boldsymbol{0}} + \underline{\boldsymbol{a}}$

1.16 Worked examples

Example 1

ABCDEF is a regular hexagon. If $\overrightarrow{AB} = \underline{\boldsymbol{a}}$ and $\overrightarrow{BC} = \underline{\boldsymbol{b}}$, express the vectors $\overrightarrow{AC}$, $\overrightarrow{AD}$, $\overrightarrow{AE}$ and $\overrightarrow{AF}$ in terms of $\underline{\boldsymbol{a}}, \underline{\boldsymbol{b}}$

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}$ ....①



By geometry AD = 2BC; AD // BC

$\therefore \overrightarrow{AD} = 2\overrightarrow{BC} = 2\underline{\boldsymbol{b}}$ ....②

$\overrightarrow{AE} = \overrightarrow{AD} + \overrightarrow{DE}$

$= 2\underline{\boldsymbol{b}} + (-\underline{\boldsymbol{a}}) = 2\underline{\boldsymbol{b}} - \underline{\boldsymbol{a}}$ ....③

By geometry BC = FE; BC // FE

$\overrightarrow{FE} = \overrightarrow{BC} = \underline{\boldsymbol{b}}$

$\overrightarrow{AF} = \overrightarrow{AE} + \overrightarrow{EF} = (2\underline{\boldsymbol{b}} - \underline{\boldsymbol{a}}) - \underline{\boldsymbol{b}} = \underline{\boldsymbol{b}} - \underline{\boldsymbol{a}}$

Example 2

The position vectors of A and B are $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$ respectively

(i) C is the midpoint of AB.

(ii) D is a point on AB such that AD : DB = 1 : 2

(iii) E is a point on AB such that AE : EB = 2 : 1

Find the position vectors of C, D and E

Let $\overrightarrow{OA} = \underline{\boldsymbol{a}}$, $\overrightarrow{OB} = \underline{\boldsymbol{b}}$. Then $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \underline{\boldsymbol{b}} - \underline{\boldsymbol{a}}$

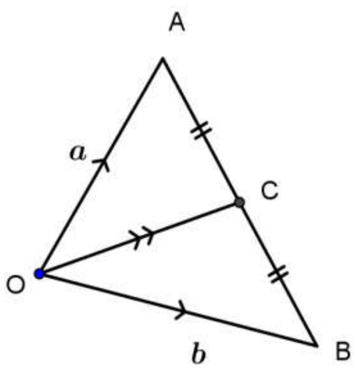
(i) AC = CB

$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$

$= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$

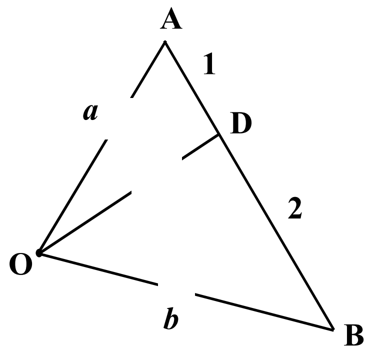
$= \underline{\boldsymbol{a}} + \frac{1}{2}(\underline{\boldsymbol{b}} - \underline{\boldsymbol{a}})$

$= \frac{1}{2}(\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}})$



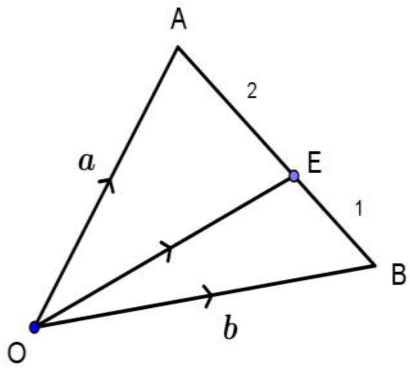
(ii) AD : DB = 1 : 2

$$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$$
$$= \overrightarrow{OA} + \frac{1}{3}\overrightarrow{AB}$$
$$= \overrightarrow{OA} + \frac{1}{3}\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$$
$$= \underline{a} + \frac{1}{3}(\underline{b} - \underline{a})$$
$$= \frac{2}{3}\underline{a} + \frac{1}{3}\underline{b} = \frac{1}{3}(2\underline{a} + \underline{b})$$



(iii)

$$\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AE}$$
$$= \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AB}$$
$$= \overrightarrow{OA} + \frac{2}{3}\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$$
$$= \underline{a} + \frac{2}{3}(\underline{b} - \underline{a})$$
$$= \frac{1}{3}\underline{a} + \frac{2}{3}\underline{b}$$
$$= \frac{1}{3}(\underline{a} + 2\underline{b})$$



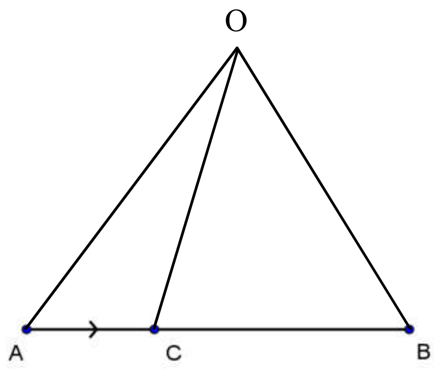
Example 3

Let $-2\underline{p} + 5\underline{q}$, $7\underline{p} - \underline{q}$ and $\underline{p} + 3\underline{q}$ be the position vectors of three points A, B and C respectively, with respect to a fixed origin O, where $\underline{p}$ and $\underline{q}$ are two non-parallel vectors. Show that the points A, B and C are collinear and find the ratio in which C divides AB.

$\overrightarrow{OA} = -2\underline{p} + 5\underline{q}$, $\quad \overrightarrow{OB} = 7\underline{p} - \underline{q}$, $\quad \overrightarrow{OC} = \underline{p} + 3\underline{q}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= (7\underline{p} - \underline{q}) - (-2\underline{p} + 5\underline{q})$$
$$= 9\underline{p} - 6\underline{q}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$$
$$= (\underline{p} + 3\underline{q}) - (-2\underline{p} + 5\underline{q})$$
$$= 3\underline{p} - 2\underline{q}$$



$$\overrightarrow{AB} = 3(3\underline{p} - 2\underline{q})$$
$$\overrightarrow{AC} = 3\underline{p} - 2\underline{q} \quad \Rightarrow \quad = \overrightarrow{AB} = 3\,\overrightarrow{AC}$$

Therefore A, B and C are collinear and AC : CB = 1 : 2

Example 4

$\underline{\boldsymbol{a}}$, $\underline{\boldsymbol{b}}$ are two non-zero and non parallel vectors and α, β are scalars. Prove that $\alpha\underline{\boldsymbol{a}} + \beta\underline{\boldsymbol{b}} = \underline{\boldsymbol{0}}$ if and only if $\alpha = 0$ and $\beta = 0$.

Assume that $\alpha = 0$ and $\beta = 0$.

$\alpha\underline{\boldsymbol{a}} + \beta\underline{\boldsymbol{b}} = \underline{\boldsymbol{0}} + \underline{\boldsymbol{0}} = \underline{\mathbf{0}}$.

Conversely let $\alpha\underline{\boldsymbol{a}} + \beta\underline{\boldsymbol{b}} = \underline{\boldsymbol{0}}$

case (i) : Suppose that $\alpha = 0$

Then $\underline{\boldsymbol{0}} + \beta\underline{\boldsymbol{b}} = \underline{\boldsymbol{0}}$

$\beta\underline{\boldsymbol{b}} = \underline{\boldsymbol{0}}$

since $\underline{\boldsymbol{b}} \neq \underline{\boldsymbol{0}}$, it follows that $\beta = 0$

If $\alpha = 0$, then $\beta = 0$

Similarly we can show that if $\beta = 0$, then $\alpha = 0$

case (ii) : Suppose that $\alpha \neq 0$

$\alpha\underline{\boldsymbol{a}} + \beta\underline{\boldsymbol{b}} = \underline{\boldsymbol{0}}$

$\alpha\underline{\boldsymbol{a}} = -\beta\underline{\boldsymbol{b}}$

$\underline{\boldsymbol{a}} = -\dfrac{\beta}{\alpha}\underline{\boldsymbol{b}} \qquad (\alpha \neq \underline{\boldsymbol{0}})$

The above equation implies that $\underline{\boldsymbol{a}} \,//\, \underline{\boldsymbol{b}}$

This is a contradiction

Hence $\alpha = 0$ and from the first part $\beta = 0$

ie. $\alpha\underline{\boldsymbol{a}} + \beta\underline{\boldsymbol{b}} = \underline{\boldsymbol{0}}$ if and only if $\alpha = 0$, $\beta = 0$

Example 5

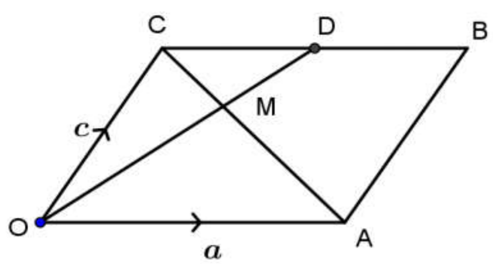
OABC is a parallelogram. D is the midpoint of BC. OD and AC intersect at M. Given that $\overrightarrow{OA} = \underline{\boldsymbol{a}}$, $\overrightarrow{OC} = \underline{\boldsymbol{c}}$

(i) Find $\overrightarrow{OD}$ interms of $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{c}}$

(ii) If OM : MD = $\lambda : 1$, find $\overrightarrow{OM}$ in terms of $\underline{\boldsymbol{a}}$, $\underline{\boldsymbol{c}}$ and λ

(iii) If AM : MC = $\mu : 1$, find $\overrightarrow{AM}$ in terms of $\underline{\boldsymbol{a}}$, $\underline{\mathbf{c}}$ and μ and hence find $\overrightarrow{OM}$

(iv) Using the results obtained in (ii) and (iii) above find the values of λ and μ.



(i) $\overrightarrow{OA} = \underline{a}$, $\overrightarrow{OC} = \underline{c}$; $\overrightarrow{OA} = \overrightarrow{CB} = \underline{a}$

$\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD}$

$= \overrightarrow{OC} + \frac{1}{2}\overrightarrow{CB}$

$= c + \frac{1}{2}a$ .... ①

OM : MD = λ : 1, $\overrightarrow{OM} = \frac{\lambda}{\lambda+1}\ \overrightarrow{OD} = \frac{\lambda}{\lambda+1}\left[\frac{1}{2}\underline{a}+\underline{c}\right]$ .... ②

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= \underline{c} - \underline{a}$

AM : MC = μ : 1, $\overrightarrow{AM} = \frac{\mu}{\mu+1}\ \overrightarrow{AC} = \frac{\mu}{\mu+1}(\underline{c} - \underline{a})$

$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = \underline{a} + \frac{\mu}{\mu+1}(\underline{c} - \underline{a})$

$= \left(1-\frac{\mu}{\mu+1}\right)\underline{a} + \frac{\mu}{\mu+1}\underline{c}$

$= \frac{1}{\mu+1}\underline{a} + \frac{\mu}{\mu+1}\underline{c}$ .... ③

From ② and ③

Since $\underline{a}$ is not parallel to $\underline{c}$

$\frac{\lambda}{2(\lambda+1)} = \frac{1}{\mu+1}$ .... ④

$\frac{\lambda}{\lambda+1} = \frac{\mu}{\mu+1}$ .... ⑤

$\frac{④}{⑤}$ gives $\frac{1}{2} = \frac{1}{\mu}$, μ = 2

If μ = 2, from ④ $\frac{\lambda}{2(\lambda+1)} = \frac{1}{3}$

λ = 2

λ = 2 = μ

ie. OM : MD = AM : MC = 2 : 1

1.17 Exercises

1. ABCDEF is a reqular hexagon. $\overrightarrow{AB} = \underline{\boldsymbol{a}}$, $\overrightarrow{AC} = \underline{\boldsymbol{b}}$ Find $\overrightarrow{AD}$, $\overrightarrow{AE}$, $\overrightarrow{AF}$ in terms of $\underline{\boldsymbol{a}}, \underline{\boldsymbol{b}}$.

2. ABCDEF is a regular hexagon and O is its centre. If $\overrightarrow{OA} = \underline{\boldsymbol{a}}$, $\overrightarrow{OB} = \underline{\boldsymbol{b}}$ find $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DE}$, $\overrightarrow{EF}$, $\overrightarrow{FA}$ in terms of $\underline{\boldsymbol{a}}, \underline{\boldsymbol{b}}$

3. ABCD is a plane quadrilatelral and O is apoint in the plane of the quadrilateral. If $\overrightarrow{AO} + \overrightarrow{CO} = \overrightarrow{DO} + \overrightarrow{BO}$, show that ABCD is a parallelogram.

4. ABC is an isosceles triangle with BA = BC and D is the midpoint of AC. Show that $\overrightarrow{BA} + \overrightarrow{BC} = 2\overrightarrow{BD}$.

5. $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$ are two vectors perpendicular to each other. Using triangle of law of vector addition show that $|\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}| = |\underline{\boldsymbol{a}} - \underline{\boldsymbol{b}}|$. When $|\underline{\boldsymbol{a}} - \underline{\boldsymbol{b}}| = 5$ and $|\underline{\boldsymbol{a}}| = 3$, find $|\underline{\boldsymbol{b}}|$.

6. $\underline{\boldsymbol{a}}, \underline{\boldsymbol{b}}$ are two vectors. such that $|\underline{\boldsymbol{a}}| = 6$, $|\underline{\boldsymbol{b}}| = 6$ and the angle between $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$ is 60°. Find $|\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}|$ and $|\underline{\boldsymbol{a}} - \underline{\boldsymbol{b}}|$.

Use vectors to prove the following questions (7,8,9).

7. ABC is a triangle. D and E are the midpoints of AB and AC. Prove that DE = $\frac{1}{2}$ BC and DE is parallel to BC.

8. ABCD is a quadrilateral. P, Q, R and S are the midpoints of AB, BC, CD and DA respectively. Show that PQRS is a parallelogram.

9. ABC is a triangle. The position vectors of A, B and C are $\underline{\boldsymbol{a}}, \underline{\boldsymbol{b}}, \underline{\boldsymbol{c}}$ respectively. Find the position vector of the centroid of the triangle ABC.

10. OABC is a parallelogram. D is the midpoint of AB. OD and AC intersects at E. $\overrightarrow{OA} = \underline{\boldsymbol{a}}$, $\overrightarrow{OB} = \underline{\boldsymbol{b}}$, OE : ED = λ : 1, CE : EA = μ : 1.

    i. Find $\overrightarrow{OD}$ in terms of $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$. Hence write the vector $\overrightarrow{OE}$ in terms of λ, $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$.

    ii. Find the vector $\overrightarrow{AC}$ and write the vector $\overrightarrow{OE}$ in terms of μ, $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$.

    iii. Using the results obtained in (i) and (ii) above find λ and μ.

    iv. When OD and CB produced meet at H, find $\overrightarrow{OH}$.

11. Let OABC be a quadrilateral and let D and E be the midpoints of the diagonal OB and AC respectively. Also let F be the mid-point of DE. By taking the position vectors of the points A, B and C with respect to O be $\underline{\boldsymbol{a}}, \underline{\boldsymbol{b}}$ and $\underline{\boldsymbol{c}}$ respectively, show that $\overrightarrow{OF} = \frac{1}{4}(\underline{a} + \underline{b} + \underline{c})$.

    Let P and Q be the midpoints of the sides OA and BC respectively. Show that the points. Show that the points P, F and Q are collinear and find the ratio PF : FQ.

12. Let A and B two distinct points not collinear with a point O. The position vectors of A and B with respect to the point O is $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$. If D is the point on AB such that BD = 2DA.

Show that the position vector of D with respect to point O is $\frac{1}{3}(2\underline{\boldsymbol{a}}+\underline{\boldsymbol{b}})$.

If $\overrightarrow{BC} = K\underline{\boldsymbol{a}}$ (K > 1) and the points O, D and C are collinear, find the value of **K** and the ratio OD : DC Express $\overrightarrow{AC}$ in terms of $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$.
Further if the line through O parallel to AC meets AB at E show that 6DE = AB.

13. Let ABCD is a trapezium such that $\overrightarrow{DC} = \frac{1}{2}\overrightarrow{AB}$. Also let $\overrightarrow{AB} = \underline{\boldsymbol{p}}$ and $\overrightarrow{AD} = \underline{\boldsymbol{q}}$. The point E lies on BC such that $\overrightarrow{BE} = \frac{1}{3}\overrightarrow{BC}$. The point of intersection F of AE and BD satisfies $\overrightarrow{BF} = \lambda\overrightarrow{BD}$. where λ $(0 < \lambda < 1)$ is a constant. Show that $\overrightarrow{AF} = (1-\lambda)\underline{\boldsymbol{p}}+\lambda\underline{\boldsymbol{q}}$.
Hence find the value of λ.

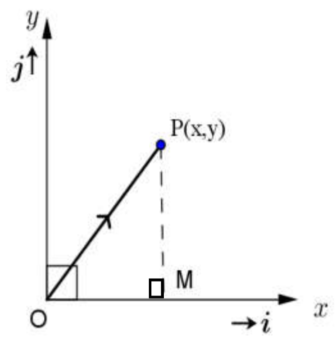
1.18 Cartesian vector notation

Consider the cartesian plane ***xoy.***
Let the unit vector in the direction Ox be $\boldsymbol{i}$, the unit vector in the direction Oy be $\boldsymbol{j}$, and P $\equiv (\boldsymbol{x}, \boldsymbol{y})$

Let $\overrightarrow{OP} = \underline{\boldsymbol{r}}$

$\underline{\boldsymbol{r}} = \overrightarrow{OP} = \overrightarrow{OM}+\overrightarrow{MP} = x\underline{\boldsymbol{i}} + y\underline{\boldsymbol{i}}$ $\quad$ $(|\overrightarrow{OM}| = OM = x,\ |\overrightarrow{MP}| = MP = y)$

$|\underline{\boldsymbol{r}}| = OP = \sqrt{x^2+y^2}$



Let $\underline{a} = a_1\underline{i} + a_2\underline{j}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j}$

$$\underline{a}+\underline{b} = (a_1\underline{i}+b_1\underline{i}) + (a_2\underline{j}+b_2\underline{j})$$
$$= (a_1+b_1)\underline{i} + (a_2+b_2)\underline{j}$$

Proof

$\overrightarrow{OA} = a_1\underline{i} + a_2\underline{j}$ $\qquad$ $A \equiv (a_1, a_2)$

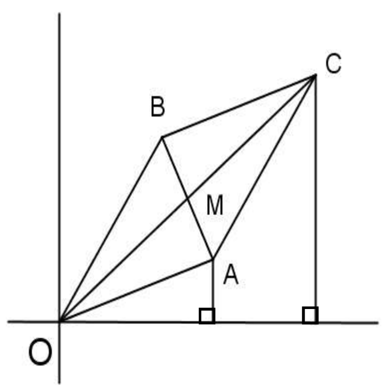
$\overrightarrow{OB} = b_1\underline{i} + b_2\underline{j}$ $\qquad$ $B \equiv (b_1, b_2)$

OACB is a parallelogram $\overrightarrow{OC} = \overrightarrow{OA}+\overrightarrow{AC} = \underline{a}+\underline{b}$

Since M is the midpoint of AB, $M \equiv \left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}\right)$

M is the midpoint of OC. $\qquad$ $C \equiv (a_1+b_1, a_2+b_2)$

$$\overrightarrow{OC} = (a_1+b_1)\underline{i} + (a_2+b_2)\underline{i}$$



If $\underline{\boldsymbol{a}} = a_1 i + a_2 j$ and $\underline{b} = b_1 i + b_2 \underline{i}$ then

$\underline{a} - \underline{b} = a + (-b) = (a_1\underline{i} + a_2\underline{i}) + (-b_1\underline{i} - b_2\underline{i})$

$\underline{a} - \underline{b} = a + (-b) = (a_1\underline{i} + a_2\underline{i}) + (-b_1\underline{i} - b_2\underline{i})$

$= (a_1 - b_1)\underline{i} + (a_2 - b_2)\underline{i}$

Example 6

If A ≡ (2, -1) and B ≡ (5, 3) find

i. $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{AB}$ in terms of $\underline{i}, \underline{j}$

ii. $|\overrightarrow{OA}|$, $|\overrightarrow{OB}|$, $|\overrightarrow{AB}|$

iii. the unit vector in the direction $\overrightarrow{AB}$

A ≡ (2, -1), B ≡ (5, 3)

(i) $\overrightarrow{OA} = 2\underline{i} - \underline{j}$ $\qquad$ $\overrightarrow{OB} = 5\underline{i} + 3\underline{j}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (5\underline{i} + 3\underline{j}) - (2\underline{i} - \underline{j})$

$= 3\underline{i} + 4\underline{j}$

(ii) $\overrightarrow{OA} = 2\underline{i} - \underline{j}$, $\qquad$ $|\overrightarrow{OA}| = \sqrt{2^2 + 1^2} = \sqrt{5}$

$\overrightarrow{OB} = 5\underline{i} + 3\underline{j}$, $\qquad$ $|\overrightarrow{OB}| = \sqrt{5^2 + 3^2} = \sqrt{34}$

$\overrightarrow{AB} = 3\underline{i} + 4\underline{j}$, $\qquad$ $|\overrightarrow{AB}| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

(iii) unit vector in the direction $\overrightarrow{AB}$ is

$$\frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \frac{1}{5}(3\underline{i} + 4\underline{j})$$

1.19 Exercises

1. Let $\underline{a} = \underline{i} - 2\underline{j}$, $\qquad b = 4\underline{j}$, $\qquad \underline{c} = 3\underline{i} - \underline{j}$ $\qquad$ Find

   i. (a) $2\underline{a} + \underline{b}$ (b) $\underline{a} + 3\underline{c}$ (c) $2\underline{a} - \underline{b} - \underline{c}$

   ii. (a) $|2\underline{a} + \underline{b}|$ (b) $|\underline{a} + 3\underline{c}|$ (c) $|2\underline{a} - \underline{b} - \underline{c}|$

   iii. the unit vector in the direction $\underline{a} + \underline{b} + \underline{c}$

2. Given that A ≡ (4, 3), B ≡ (6, 6) and C ≡ (0, 1)

   (a) Write the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$

   (b) Find $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$

   (c) Find $|\overrightarrow{AB}|$, $|\overrightarrow{BC}|$, $|\overrightarrow{CA}|$

3. O is the origin and $\overrightarrow{OA} = -\underline{i} + 5\underline{j}$, $\qquad \overrightarrow{OB} = 2\underline{i} + 4\underline{j}$ and $\overrightarrow{OC} = 2\underline{j}$. Find $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$. Hence show that ABC is an isosceles triangle.

4. If $\overrightarrow{OA} = \underline{i} + 2\underline{j}$, $\overrightarrow{OB} = 3\underline{i} - \underline{j}$ and $\overrightarrow{OC} = -\underline{i} + 5\underline{j}$. Find $\overrightarrow{AB}$ and $\overrightarrow{CA}$ and hence show that the points A, B and C are collinear.

5. The position vectors of the points A and B are $\underline{a}$ and $\underline{b}$ respectively, where $\underline{a} = 2\underline{i} + 3\underline{j}$ *and* $\underline{b} = \underline{i} + 5\underline{j}$.
   (i) If R is the midpoint of AB, find the position vector of R interms of $\underline{i}, \underline{j}$.
   (ii) IF $\underline{c} = 2\underline{a} - \underline{b}$. Find the unit vector along $\underline{c}$ interms of $\underline{i}, \underline{j}$

6. (a) Find in the form $a\underline{i} + b\underline{j}$, a vector of magnitude 10 units in the direction $3\underline{i} - 4\underline{j}$
   (b) A $\equiv$ (-2, -5) and B $\equiv$ (3, 7)
   (i) Write $\overrightarrow{OA}$, $\overrightarrow{OB}$ and hence $\overrightarrow{AB}$
   (ii) Find in the form $a\underline{i} + b\underline{j}$, a vector of magnitude 65 units in the diretion $\overrightarrow{AB}$

1.20 Scalar product of two vectors

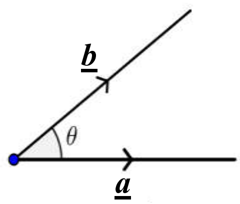
We learnt earlier the rules of vector addition and subtraction. Two types of products have been defined.

(i) The scalar product of two vectors.
(ii) The vector product of two vectors.

The scalar product is also known as the dot product. The result of a dot products a scalar and the result of the vector product is a vector.

Definition : **Scalar Product**

Let $\underline{a}$ and $\underline{b}$ be any two non-zero vectors and θ be the angle between the two vectors. The scalar product of two vectors $\underline{a}$ and $\underline{b}$ is dfined as $\underline{a} \bullet \underline{b} = |\underline{a}|\ |\underline{b}| \cos\theta$ ($0 \le \theta \le \pi$)



Properties of the scalar product

1. $\underline{a} \bullet \underline{b} = \underline{b} \bullet \underline{a}$ (Commutative Law)

By definition, $\underline{a} - \underline{b} = |\underline{a}|\ |\underline{b}| \cos\theta$

$= |\underline{b}|\ |\underline{a}| \cos\theta$

Hence $\underline{a} \bullet \underline{b} = \underline{b} \bullet \underline{a}$

2. If $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$ are two non - zero vectors $\underline{\boldsymbol{a}} \bullet \underline{\boldsymbol{b}} = \boldsymbol{0}$. if and only if $\underline{\boldsymbol{a}}$ is perpenticular to $\underline{\boldsymbol{b}}$.

$$\underline{\boldsymbol{a}} \bullet \underline{\boldsymbol{b}} = 0 \Leftrightarrow \quad |\underline{a}|\ |\underline{b}| \cos\theta = 0$$

$$\Leftrightarrow \quad \cos\theta = 0 \qquad (\underline{\boldsymbol{a}}, \underline{\boldsymbol{b}} \neq \underline{\boldsymbol{0}})$$

$$\Leftrightarrow \quad \theta = \frac{\pi}{2}$$

3. $\underline{\boldsymbol{a}} \bullet \underline{\boldsymbol{a}} = |\underline{a}|\ |\underline{a}| \cos 0 = |\underline{a}|^2$ also writen $\underline{\boldsymbol{a}}^2$

$$\underline{\boldsymbol{i}} \bullet \underline{\boldsymbol{i}} = |\underline{i}|\ |\underline{i}| \cos 0 = 1 \times 1 \times 1 = 1$$

$$\underline{\boldsymbol{j}} \bullet \underline{\boldsymbol{j}} = |\underline{j}|\ |\underline{j}| \cos 0 = 1 \times 1 \times 1 = 1$$

$$\underline{\boldsymbol{i}} \bullet \underline{\boldsymbol{j}} = |\underline{i}|\ |\underline{j}| \cos \frac{\pi}{2} = 1 \times 1 \times 0 = 0$$

i.e: $\underline{\boldsymbol{i}} \bullet \underline{\boldsymbol{i}} = \underline{\boldsymbol{j}} \bullet \underline{\boldsymbol{j}} = 1$ and $\underline{\boldsymbol{i}} \bullet \underline{\boldsymbol{j}} = \underline{\boldsymbol{j}} \bullet \underline{\boldsymbol{i}} = 0$

4. $\underline{\boldsymbol{a}}$, $\underline{\boldsymbol{b}}$, $\underline{\boldsymbol{c}}$ are vectors.

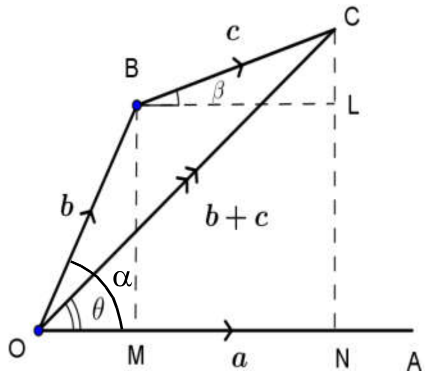
$\underline{\boldsymbol{a}} . (\underline{\boldsymbol{b}} + \underline{\boldsymbol{c}}) = \underline{\boldsymbol{a}}.\underline{\boldsymbol{b}} + \underline{\boldsymbol{a}}.\underline{\boldsymbol{c}}$ (Distributive Law)

Let the angle between

$\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$ be α

$\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{c}}$ be β

$\underline{\boldsymbol{a}}$ and $(\underline{\boldsymbol{b}} + \underline{\boldsymbol{c}})$ be θ



$$\underline{\boldsymbol{a}} \bullet (\underline{\boldsymbol{b}} + \underline{\boldsymbol{c}}) = |\underline{a}| \bullet |\underline{b} + \underline{c}| \cos\theta$$

$= (OA) \quad (OC) \cos\theta$

$= (OA) . (ON)$

$= (OA) (OM + MN)$

$= OA.OM + OA . MN$

$= OA.\ OB \cos\alpha + OA.\ BC\cos\beta \qquad (MN = BL)$

$= \underline{\boldsymbol{a}} \bullet \underline{\boldsymbol{b}} + \underline{\boldsymbol{a}} \bullet \underline{\boldsymbol{c}}$

$= \overrightarrow{OA} \cdot \overrightarrow{OB} + \overrightarrow{OA} \cdot \overrightarrow{BC}$

Therefore $\quad \underline{\boldsymbol{a}} . (\underline{\boldsymbol{b}} + \underline{\boldsymbol{c}}) = \underline{\boldsymbol{a}} \bullet \underline{\boldsymbol{b}} + \underline{\boldsymbol{a}} \bullet \underline{\boldsymbol{c}}$

5. Let $\underline{\boldsymbol{a}} = a_1 \underline{i} + a_2 \underline{j}$ and $\underline{\boldsymbol{b}} = b_1 \underline{i} + b_2 \underline{j}$

$$\underline{\boldsymbol{a}} \bullet \underline{\boldsymbol{b}} = (a_1 \underline{i} + a_2 \underline{j}) \bullet (b_1 \underline{i} + b_2 \underline{j})$$

$= a_1 \underline{i} \bullet (b_1 \underline{i} + b_2 \underline{j}) + a_2 \underline{j} \bullet (b_1 \underline{i} + b_2 \underline{j})$

$= a_1 \underline{i} \bullet b_1 \underline{i} + a_1 \underline{i} \bullet b_2 \underline{j} + a_2 \underline{j} \bullet b_1 \underline{i} + a_2 \underline{i} \bullet b_2 \underline{j}$

$= a_1 b_1 + a_2 b_2$ (since $\underline{i} \bullet \underline{i} = \underline{j} \bullet \underline{j} = 1$ and $\underline{i} \bullet \underline{j} = \underline{j} \bullet \underline{i} = 0$)

Example 7

$\underline{\boldsymbol{a}} = 2\underline{\boldsymbol{i}} - 3\underline{\boldsymbol{j}}$ and $\underline{\boldsymbol{b}} = \underline{\boldsymbol{i}} - 3\underline{\boldsymbol{j}}$ Find the angle between $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$

$$\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}} = |\underline{a}|\ |\underline{b}| \cos\theta$$

$$|\underline{a}| = \sqrt{2^2+3^2} = \sqrt{13} \qquad |\underline{b}| = \sqrt{1^2+3^2} = \sqrt{10}$$

$$\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}} = \sqrt{13} \times \sqrt{10}\ \cos\theta \qquad ①$$

$$\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}} = (2\underline{\boldsymbol{i}} - 3\underline{\boldsymbol{j}}) \cdot (\underline{\boldsymbol{i}} - 3\underline{\boldsymbol{j}})$$

$$= 2\underline{\boldsymbol{i}} \cdot (\underline{\boldsymbol{i}} - 3\underline{\boldsymbol{j}}) - 3\underline{\boldsymbol{j}} \cdot (\underline{\boldsymbol{i}} - 3\underline{\boldsymbol{j}})$$

$$= 2 + 0 - 0 + 9 = 11 \qquad ②$$

From (1) and (2) $\sqrt{130}\ \cos\theta = 11$

$$\cos\theta = \frac{11}{\sqrt{130}}, \theta = \cos^{-1}\left(\frac{11}{\sqrt{130}}\right)$$

Example 8

i. If $\underline{\boldsymbol{a}}$, $\underline{\boldsymbol{b}}$ are two vectors such that $|\underline{a}| = |\underline{b}| = |\underline{a}+\underline{b}|$,

Find the angle between $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$.

ii. Two vectors $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$ are such that $\underline{\boldsymbol{a}}$ is perpendicular to $\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}$. If $|\underline{b}| = \sqrt{2}|\underline{a}|$,

Show that $(2\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}})$ is perpendicular to $\underline{\boldsymbol{b}}$.

i. $|\underline{a}| = |\underline{b}| = |\underline{a}+\underline{b}|$

$$|\underline{a}|^2 = |\underline{a}+\underline{b}|^2$$

$\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{a}} = (\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}) \cdot (\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}})$ (by definition)

$$|\underline{a}|^2 = |\underline{a}|^2 + |\underline{b}|^2 + 2\underline{\boldsymbol{a}}.\underline{\boldsymbol{b}}$$

$$-|\underline{b}|^2 = 2|\underline{a}|\,|\underline{b}|\cos\theta$$

$$-|\underline{b}|^2 = 2|\underline{b}|\,|\underline{b}|\cos\theta$$

$$\cos\theta = -\frac{1}{2} \qquad \theta = \frac{2\pi}{3}$$

ii. $(\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}) \cdot \underline{\boldsymbol{a}} = 0 \Rightarrow \underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{a}} + \underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}} = 0$

$$|\underline{a}|^2 + \underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}} = 0 \ \text{............} \qquad ①$$

$$(2\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}) \cdot \underline{\boldsymbol{b}} = 2\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}} + \underline{\boldsymbol{b}} \cdot \underline{\boldsymbol{b}}$$

$$= 2\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}} + |\underline{b}|^2$$

$$= -2|\underline{a}|^2 + |\underline{b}|^2 \qquad \text{(from ①)}$$

$$= -2|\underline{a}|^2 + 2|\underline{a}|^2 \qquad \text{(since } |\underline{b}| = \sqrt{2}\underline{\boldsymbol{a}}\text{)}$$

$$= 0$$

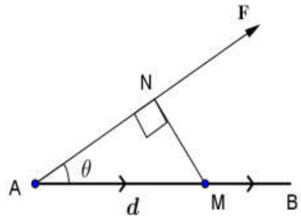
Hence $(2\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}})$ is perpenticular to $\underline{\boldsymbol{b}}$.

Example 9

If a constant force $\underline{\boldsymbol{F}}$ acting on a body moves it a distance $\boldsymbol{d}$ in the direction AB, where $\overrightarrow{AB}$ makes an angle θ with $\underline{\boldsymbol{F}}$, the work done by the force $\underline{\boldsymbol{F}}$ is $\underline{\boldsymbol{F}} \cdot \underline{\boldsymbol{d}}$.

If the point of application of a force $\underline{\boldsymbol{F}} = 2\underline{\boldsymbol{i}} + 3\underline{\boldsymbol{j}}$ makes a displacement $\underline{\boldsymbol{S}} = 5\underline{\boldsymbol{i}} - 3\underline{\boldsymbol{j}}$,

Find the workdone by the force $\underline{\boldsymbol{F}}$.

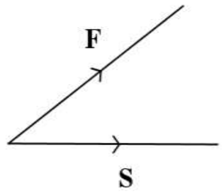


Workdone by $\underline{\boldsymbol{F}}$ is

$= \quad |\underline{F}|.AN$

$= \quad |\underline{F}|.AM\cos\theta \qquad (\overrightarrow{AM} = d)$

$= \quad \underline{\boldsymbol{F}} \cdot \underline{\boldsymbol{d}}$



Workdone by $\underline{\boldsymbol{F}} \cdot \underline{\boldsymbol{S}}$

$= \quad (2\underline{i} + 3\underline{j}) \cdot (5\underline{i} - 3\underline{j})$

$= \quad 2 \times 5 - 3 \times 3 = 10 - 9 = 1$ Joule

1.21 Exercises

1. If $\underline{a} = 3\underline{i} + \underline{j}$ and $\underline{b} = -\underline{i} + 2\underline{j}$, Find the angle between $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$.

2. If $\underline{a} = p\underline{i} + 3\underline{j}$ and $\underline{b} = 2\underline{i} + 6\underline{j}$ are two perpendcular vectors,
   i. find the value of p
   ii. find $|\underline{a}|$ and $|3\underline{b} - \underline{a}|$
   iii. find $\underline{\boldsymbol{a}} \,.\, (3\underline{\boldsymbol{b}} - \underline{\boldsymbol{a}})$
   iv. find the angle between $\underline{\boldsymbol{a}}$ and $(3\underline{\boldsymbol{b}} - \underline{\boldsymbol{a}})$

3. Two vectors $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$ are such that $|\underline{a}| = |\underline{b}| = |\underline{a} - \underline{b}|$.
   Find the angle between $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$

4. If $|\underline{a}| = 3$, $|\underline{b}| = 2$ and $|\underline{a} - \underline{b}| = 4$,
   Find (i) $\underline{\boldsymbol{a}} \,.\, \underline{\boldsymbol{b}}$
   (ii) $|\underline{a} + \underline{b}|$

5. If $\underline{\boldsymbol{a}}$ and $(\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}})$ are perpendicular vectors to each other,
   Show that $|\underline{a} + \underline{b}|^2 = |\underline{b}|^2 - |\underline{a}|^2$.

6. Using the dot product, show that the diagonals of a rhombus are perpendicular to each other.

7. Show that if $|\underline{a} + \underline{b}| = |\underline{a} - \underline{b}|$ then $\underline{\boldsymbol{a}} \,.\, \underline{\boldsymbol{b}} = 0$. Hence, show that if the diagonals of a parallelogram are equal, then it is a rectangle.

8. $\underline{a} = \underline{i} + \sqrt{3}\underline{j}$ where $\underline{\boldsymbol{i}}$ and $\underline{\boldsymbol{j}}$ have the usual meaning. $\underline{\boldsymbol{b}}$ is a vector with magnitude $\sqrt{3}$.
   If the angle between the vectors $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$ is $\frac{\pi}{3}$, Find $\underline{\boldsymbol{b}}$ in he form $\boldsymbol{x}\underline{\boldsymbol{i}} + \boldsymbol{y}\underline{\boldsymbol{j}}$ where $\boldsymbol{x}$ (<0) and $\boldsymbol{y}$ are constants to be determined.

9. AB is a diameter of a circle and P is any point on the circumference of the circle. show that APB is a right angle (use dot product)

10. Using dot product, with the usual notation prove that for any triangle ABC,
    $$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

2.0 System of coplanar forces acting on a particle

2.1 Introduction

Statics :

Statics is a branch of mechanics. It deals with bodies in equlibrium under the action of forces.

Force :

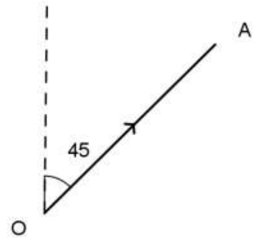
Force is defined as any cause which alters or tend to alter a body's state of rest or of uniform motion in a straight line. Unit of force is Newton and denoted by N.

To specify a force which acts on a particle it is important to give.

i. magnitude of the force

ii. direction of the force and

iii. point of its application



Force can be represented by a directed line segment. Let a force 10 Newton (N) is acting at a point O in the north - east direction. Then the force can be represented by the directed line segment OA. In this diagram the length OA represents 10 units and the arrow mark gives its direction.

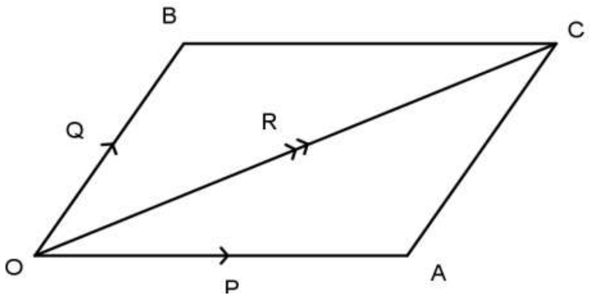
Resultant force:

When a body is acted upon by a number of forces, a single force equivalent to the given forces is called resultant force.

2.2 The parallelogram law of forces

The parallelogram law of forces is the fundamental theorem of Statics and it can be verified by experiment.

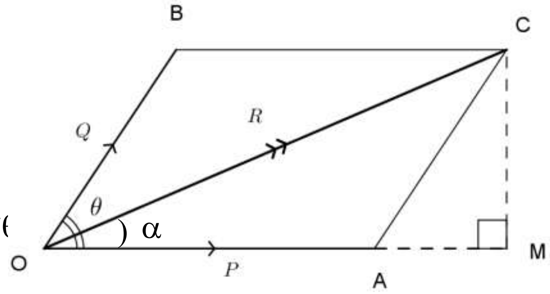
If two forces, acting on a particle at O, be represented in magnitude and direction by the two straight lines OA and OB respectively, then the resultant is represented in magnitude and direction by the diagonal OC of the parallelogram OACB.



Let the forces P and Q be represented by OA and OB respectively. Then the resultant R of P and Q is represented by the diagonal OC of the parallelogram OACB.

Let the angle between P and Q be θ and the resultant R makes an angle α with P.

By Pythagora's theorem



$$OC^2 = OM^2 + MC^2$$

$$= (OA + AM)^2 + MC^2$$

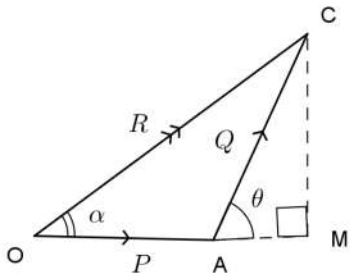
$$R^2 = (P + Q\cos\theta)^2 + (Q\sin\theta)^2$$

$$= P^2 + 2PQ\cos\theta + Q^2\cos^2\theta + Q^2\sin^2\theta$$

$$R^2 = P^2 + Q^2 + 2PQ\cos\theta$$

$$\tan\alpha = \frac{CM}{OM} = \frac{CM}{OA+AM}$$

$$= \frac{Q\sin\theta}{P+Q\cos\theta}$$

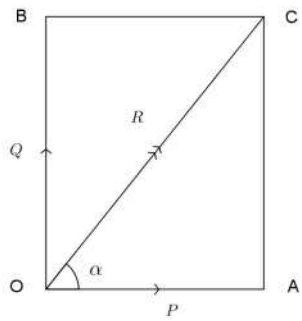


$$\boxed{\begin{aligned} R^2 &= P^2 + Q^2 + 2PQ\cos\theta \\ \tan\alpha &= \frac{Q\sin\theta}{P+Q\cos\theta} \end{aligned}}$$

When $\theta = 90°$, $\cos\theta = \cos 90 = 0$; $\sin\theta = \sin 90 = 1$

$$R^2 = P^2 + Q^2, \text{ and } \tan\alpha = \frac{Q}{P}$$

when $Q = P$



$$R^2 = P^2 + P^2 + 2P \times P \times \cos\theta$$

$$= 2P^2 + 2P^2\cos\theta = 2P^2(1 + \cos\theta)$$

$$= 2P^2 \times 2\cos^2\frac{\theta}{2} = 4P^2\cos^2\frac{\theta}{2}$$

$$R = 2P.\cos\frac{\theta}{2}$$

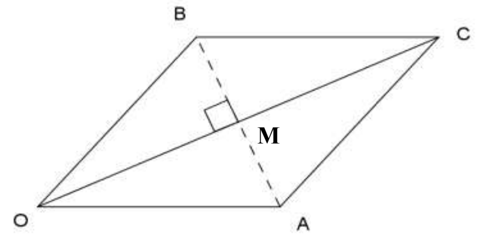
$$\tan\alpha = \frac{P\sin\theta}{P+P\cos\theta} = \frac{\sin\theta}{1+\cos\theta} = \frac{2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

$$\alpha = \frac{\theta}{2}$$

When the two forces are equal the resultant of these two forces bisects the angle between the forces.

Alternate Method (Using geometry)

When P = Q; OA = OB



The parallelogram OACB is rhombus.

i. OC and AB intersect at right angles.

ii. $\angle AOC = \angle BOC \quad (= \frac{\theta}{2})$

$$OC = 2OM = 2OA\cos\frac{\theta}{2}$$

$$R = 2P\cos\frac{\theta}{2}$$

Example 1:

Forces 3P and 5P act at a point and the angle between the forces is 60°. Find the resultant.

$R^2 = P^2 + Q^2 + 2PQ\cos\theta$

$= (3P)^2 + (5P)^2 + 2 \times 3P \times 5P \cdot \cos 60°$

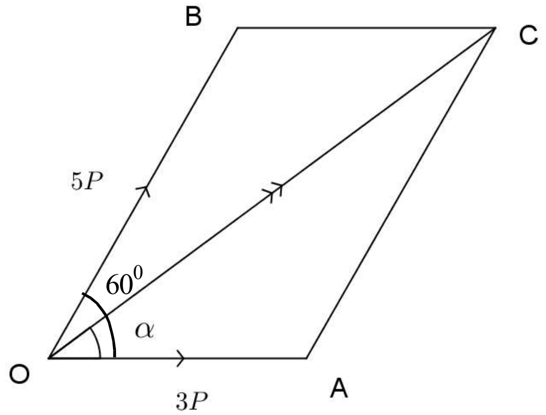
$= 9P^2 + 25P^2 + 15P^2 = 49P^2$

$R = 7P$

$$\tan\alpha = \frac{5P\sin 60°}{3P + 5P\cos 60°}$$

$$\tan\alpha = \frac{5\sqrt{3}}{11}$$

$$\alpha = \tan^{-1}\left(\frac{5\sqrt{3}}{11}\right)$$



Example 2

The resultant of two forces 8P and 5P acting at a point is 7P. Find the angle between these two forces.

Let Q be the angle between the forces 8P and 5P.

$$R^2 = P^2 + Q^2 + 2PQ\cos\theta$$

$$(7P)^2 = (8P)^2 + (5P)^2 + 2 \times 8P \times 5P\cos\theta$$

$$49P^2 = 64P^2 + 25P^2 + 80P^2\cos\theta$$

$$-40 = 80\cos\theta$$

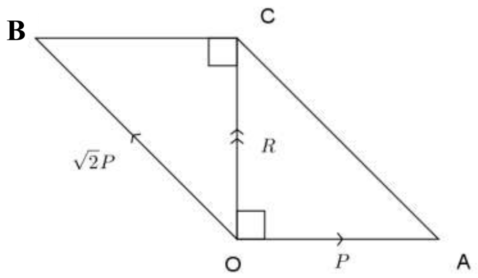
$$\cos\theta = -\frac{1}{2}$$

$$\theta = 120°$$

Example 3 :

The resultant of two forces P and $\sqrt{2}P$, acting at a point is at right angle to the smaller force. Find the resultant and the angle between the two given forces.

Applying Pythagora's theorem,



$$OC^2 + CB^2 = OB^2$$

$$R^2 + P^2 = (\sqrt{2}P)^2$$

$$R^2 = P^2;\ R = P;$$

Therefore OC = BC and $\angle BOC = 45°$

The angle between the force is $90° + 45° = 135°$

2.3 Resolution of a force into two directions

a. Rectangular resolution of a force

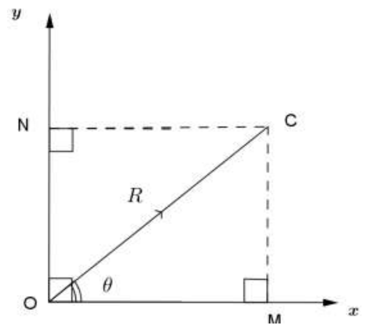
We have studied that two forces acting at a point can be reduced to an equivalent single force (resultant) using parallelogram of forces. Conversely a single force can be resolved into pair of component in an infinite number of ways.

Let R be a force acting on a particle. It can be resolved in two perpendicular directions.

Let the force R be represeated by OC.

We have to resolve the force R, along Ox and Oy.

Let θ be the angle R makes with Ox axis.

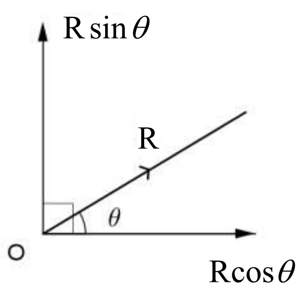


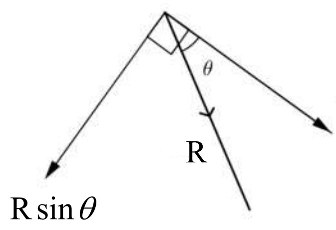
OMCN is a rectangle.

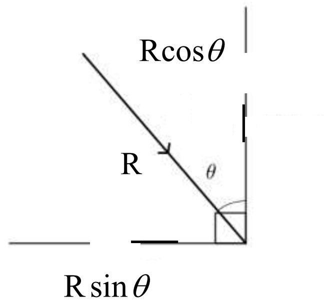
$$\cos\theta = \frac{OM}{OC}, \quad OM = OC\cos\theta = R\cos\theta$$

$$\sin\theta = \frac{MC}{OC}, \quad MC = OC\sin\theta = R\sin\theta = ON$$

Hence the resolved components of R along Ox and Oy are $R\cos\theta$ and $R\sin\theta$ respectively.

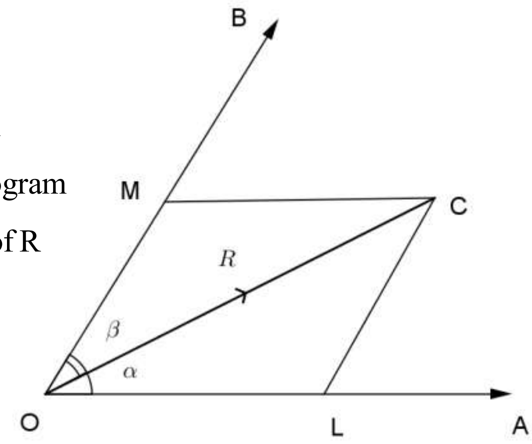






b. Oblique Resolution

Let R be the given force and let OA and OB be the given directions along which the force R is to be resolved.



R is represented by OC.

Through C, draw lines CM and CL parallel to OA and OB respectively. Now, OLCM is a parallelogram

Hence OL and OM are the resolved components of R along OA and OB respectively.

Let $C\hat{O}A = \alpha$ and $C\hat{O}B = \beta$

Using sine law in the triangle OLC.

$$\frac{OL}{\sin O\hat{C}L} = \frac{LC}{\sin C\hat{O}L} = \frac{OC}{\sin O\hat{L}C}$$

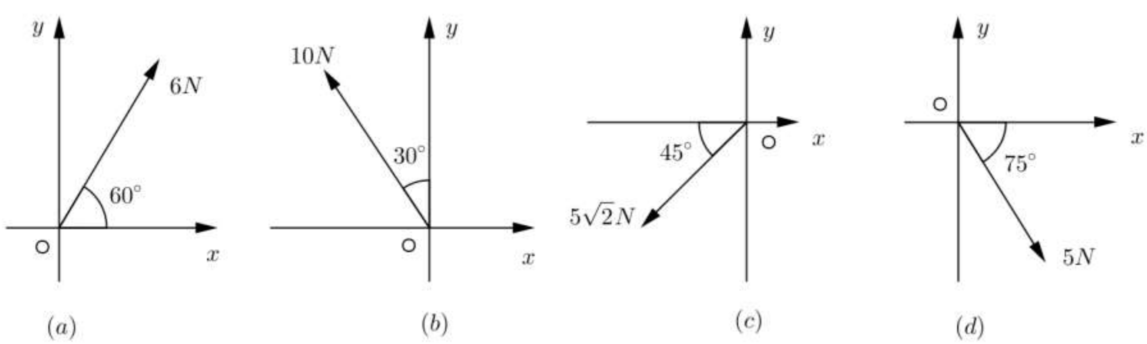
$$\frac{OL}{\sin\beta} = \frac{LC}{\sin\alpha} = \frac{R}{\sin\left[180-(\alpha+\beta)\right]}$$

$$\frac{OL}{\sin\beta} = \frac{LC}{\sin\alpha} = \frac{R}{\sin(\alpha+\beta)}$$

$$OL = \frac{R\sin\beta}{\sin(\alpha+\beta)}, \qquad LC = \frac{R\sin\alpha}{\sin(\alpha+\beta)}$$

Hence the resolved components along OA, OB are $\frac{R\sin\beta}{\sin(\alpha+\beta)}, \frac{R\sin\alpha}{\sin(\alpha+\beta)}$ respectively.

Example 4



(a) $\rightarrow$ X = $6\cos 60^0$ $= 6 \times \frac{1}{2}$ $= 3N$

$\uparrow$ Y = $6\sin 60$ $= 6 \times \frac{\sqrt{3}}{2}$ $= 3\sqrt{3}$ N

(b) $\leftarrow$ X = $10\sin 30$, $= 10 \times \frac{1}{2} = 5N$

$\uparrow$ Y = $10\cos 30 = 10 \times \frac{\sqrt{3}}{2} = 5\sqrt{3}$ N

(c) $\leftarrow$ X = $5\sqrt{2}\cos 45 = 5\sqrt{2} \times \frac{1}{\sqrt{2}} = 5N$

$\downarrow$ Y = $5\sqrt{2}\sin 45 = 5\sqrt{2} \times \frac{1}{\sqrt{2}} = 5N$

(d) $\rightarrow$ X = $5\cos 75 = 5\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)$N

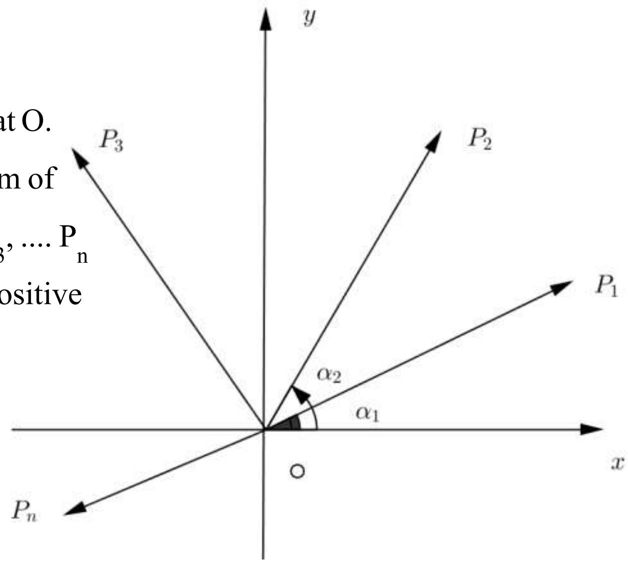
$\downarrow$ Y = $5\sin 75 = 5\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)$N

2.4 Resultant of a system of coplanar forces acting at a point

Let Ox and Oy be two perpendicular axes.

In the plane of xoy, a system of forces act at O.

Let $P_1, P_2, P_3, \ldots\ldots P_n$ be a coplanar system of forces acting at O and the forces $P_1, P_2 P_3, \ldots. P_n$ makes angles $\alpha_1, \alpha_2, \alpha_3, \ldots.. \alpha_n$ with the positive direction of Ox axis.



Resolving along Ox,

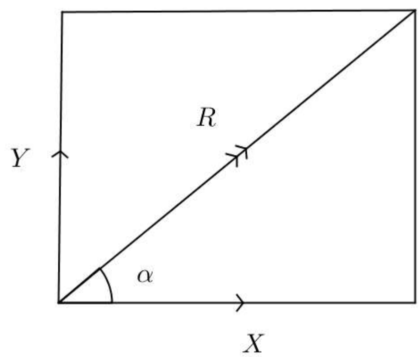
$$\rightarrow X = P_1\cos\alpha_1 + P_2\cos\alpha_2 + P_3\cos\alpha_3 + \ldots\ldots\ldots + P_n\cos\alpha_n$$

$$\uparrow Y = P_1\sin\alpha_1 + P_2\sin\alpha_2 + P_3\sin\alpha_3 + \ldots\ldots\ldots + P_n\sin\alpha_n$$

If R is the resultant, then

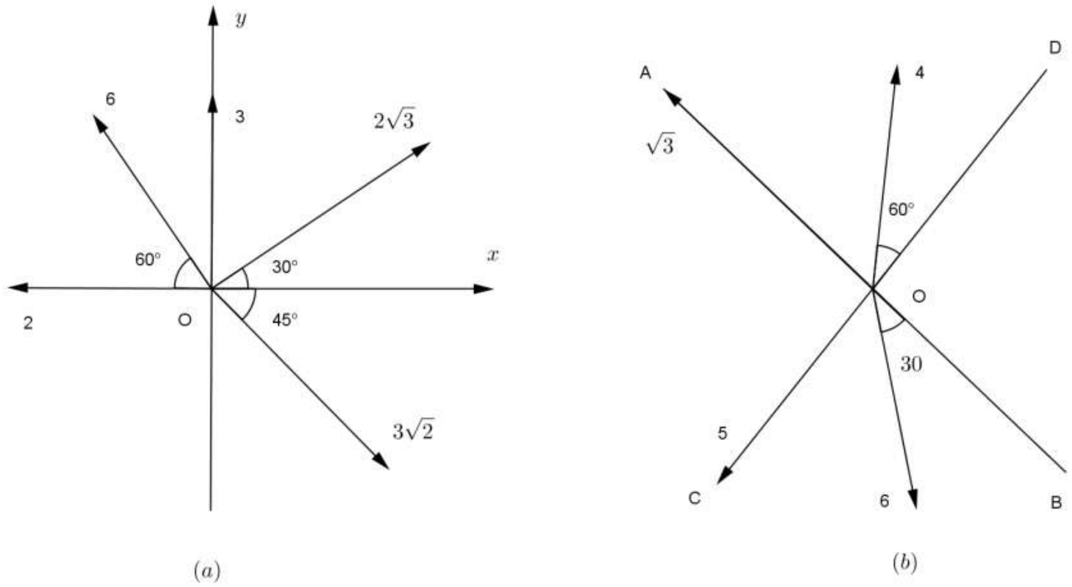
$$R = \sqrt{X^2 + Y^2}$$

$$\tan\alpha = \frac{Y}{X}$$



Example 5

For each of the following sets of forces acting at O, find the resultant.



Ox and Oy are perpendicular to each other AB and CD are perpendicular to each other

(a)   Resolving along Ox,

$X = 2\sqrt{3}\cos 30 - 6\cos 60 - 2 + 3\sqrt{2}\cos 45$

$= 3 - 3 - 2 + 3 = 1$

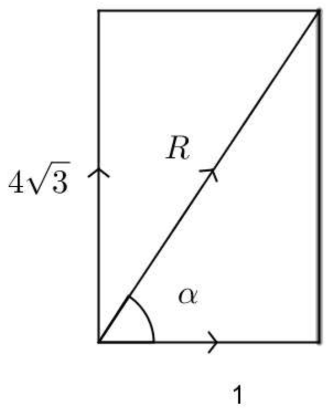
Resolving along Oy,

$Y = 3 + 2\sqrt{3}\sin 30 + 6\sin 60 - 3\sqrt{2}\sin 45$

$= 3 + \sqrt{3} + 3\sqrt{3} - 3 = 4\sqrt{3}$

$R^2 = X^2 + Y^2 = \left(4\sqrt{3}\right)^2 + 1^2 = 49$

$R = 7N, \tan\alpha = 4\sqrt{3}$



(b)   Resolving along BA

$X = \sqrt{3} + 4\sin 60 - 6\cos 30$

$= \sqrt{3} + 2\sqrt{3} - 3\sqrt{3} = 0$

Resolving along DC

$Y = 5 - 4\cos 60 + 6\sin 30$

$= 5 - 2 + 3 = 6$

Hence the resultant is 6N along DC

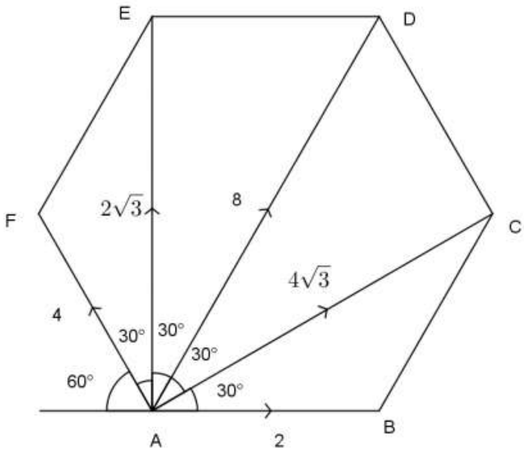
Example 6

ABCDEF is a regular hexagon. Forces of magnitudes 2, $4\sqrt{3}$, 8, $2\sqrt{3}$ and 4 newtons act at A in the directions AB, AC, AD, AE and AF respectively. Find the resultant.

$B\hat{A}E = 90°$

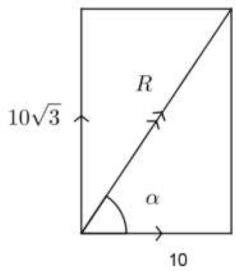
Take AB and AE are x and y axis respectively.

Resolving along AB



$$X = 2 + 4\sqrt{3}\cos 30 + 8\cos 60 - 4\cos 60$$
$$= 2 + 6 + 4 - 2 = 10$$

$$Y = 4\sqrt{3}\sin 30 + 8\sin 60 + 2\sqrt{3} + 4\sin 60$$
$$= 2\sqrt{3} + 4\sqrt{3} + 2\sqrt{3} + 2\sqrt{3} = 10\sqrt{3}$$



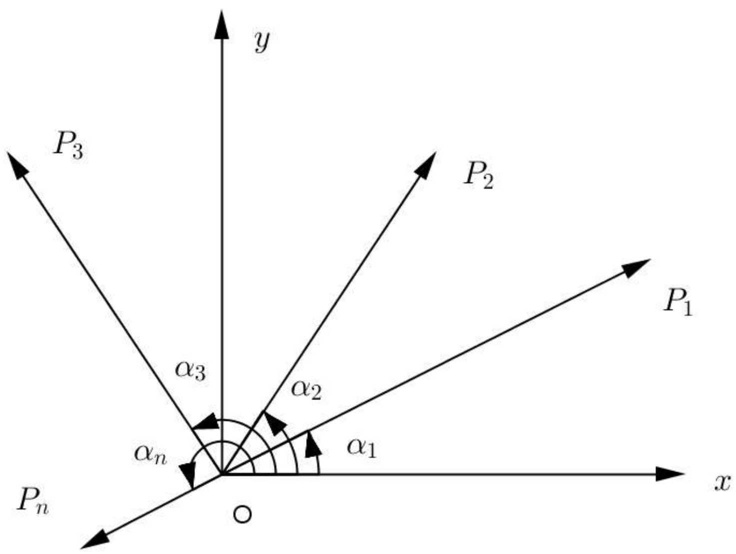
$$R^2 = X^2 + Y^2 = 10^2 + (10\sqrt{3})^2$$
$$= 400$$
$$R = 20N$$

$$\tan\alpha = \frac{10\sqrt{3}}{10} = \sqrt{3}; \qquad \alpha = 60°$$

Hence the resultant is 20N and act along AD.

2.5 Equilibrium of coplanar forces acting at a point

Let Ox and Oy be two perpendicular axes. In the plane of xOy, a system of forces act at O.



Let P_1, P_2, P_3 ..... P_n be a set of coplanar forces acting at O and the forces P_1, P_2, P_3, ... P_n make angles α_1, α_2, α_3, ... α_n with the positive direction of Ox axis.

Resolving along Ox axis

$$\rightarrow X = P_1\cos\alpha_1 + P_2\cos\alpha_2 + ... + P_n\cos\alpha_n$$

$$\uparrow Y = P_1\sin\alpha_1 + P_2\sin\alpha_2 + ... + P_n\sin\alpha_n$$

$$R = X^2 + Y^2$$

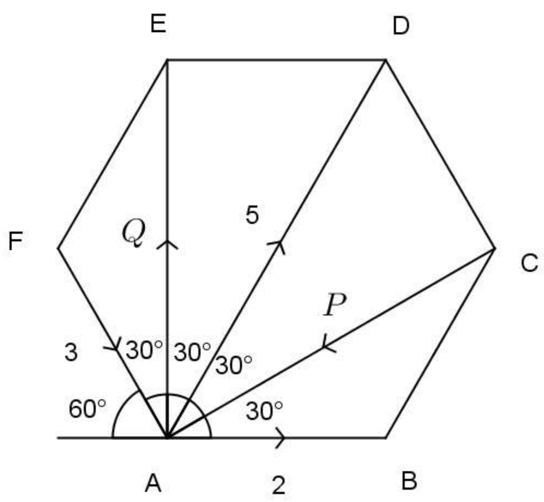
Since the particle is an equilibrium, the resultant force R = O.

$$R = 0 \quad \Rightarrow \quad X = 0, \quad Y = 0 \quad (\text{since } X^2 \geq 0, Y^2 \geq 0)$$

* It is necessary that the resolved components of the forces acting on the particle, in two differnt direction must be zero.

Example 7

ABCDEF is a regular hexagon. Forces of magni
CA, AD, AE and FA respectively. Find P and Q i



Resolving along AB

$$X = 2 - P\cos30 + 5\cos60 + 3\cos60$$

$$= 2 - \frac{\sqrt{3}P}{2} + \frac{5}{2} + \frac{3}{2}$$

$$= 6 - \frac{\sqrt{3}P}{2}$$

$$Y = Q - P\sin30 + 5\sin60 - 3\sin60$$

$$= Q - \frac{P}{2} + \sqrt{3}$$

Since the system of forces is in equilibrium

$$X = 0, \quad Y = 0$$

$$X = 0 \quad \Rightarrow \quad 6 - \frac{P\sqrt{3}}{2} = 0; \quad P = \frac{12}{\sqrt{3}} = 4\sqrt{3}\text{ N}$$

$$Y = 0 \quad \Rightarrow \quad Q - \frac{P}{2} + \sqrt{3} = 0$$

$$Q - 2\sqrt{3} + \sqrt{3} = 0$$

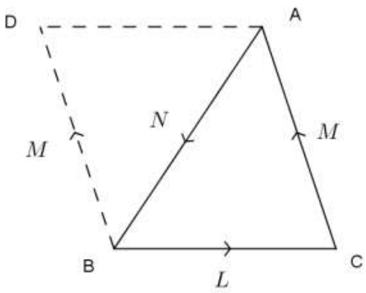
$$Q = \sqrt{3}\text{N}$$

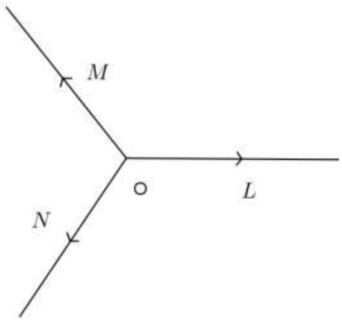
2.6 Three coplanar forces acting on a particle

1. Triangle of forces

If three forces, acting on a particle, can be represented in magnitude and direction by the sides of a triangle taken in order the forces will be in equilibrium.

Let L, M, N be three forces acting at O and represented by BC, CA, AB respectively (in magnitude and direction) taken in order of a triangle ABC, then L, M, N are in equilibrium.





Complete the parallelogram BCAD.

BD = CA, BD || CA

BD represents M in magnitude and direction.

Using parallelogram law of forces, the resultant R of L and M is represented by $\overrightarrow{BA}$.

i.e : BA = N and direction of R is opposite to N.

Since R = N and opposite in direction and act at O

Hence L, M, N are in equilibrium.

OR

Using vectors $\overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{BA}$

$$\left(\overrightarrow{BC}+\overrightarrow{CA}\right)+\overrightarrow{AB} = \overrightarrow{BA}+\overrightarrow{AB} = \underline{\boldsymbol{0}}$$

The vector sum of three forces is O. Since all three forces act at a point, they are in equilibrium.

2. Converse of triangle of forces

If three forces acting at a particle are in equilibrium. then they can be represented in magnitude and direction by the three sides of a triangle taken in order.

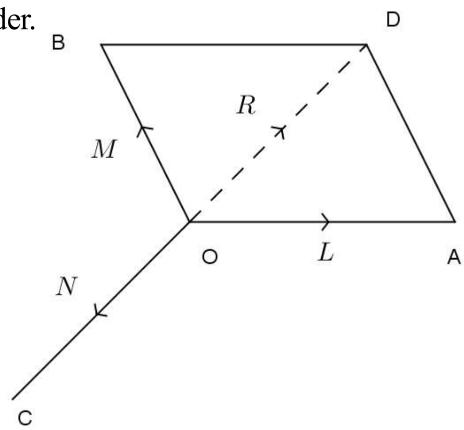
Let L, M and N be three forces acting at a particle

and they are in equilibrium.

Let the three forces L, M, N acting at O

are represented by OA, OB, OC

(in magnitude and direction) respectively.



Complete the parallelogram OADB. Using parallelogram law of forces the resultant R of L and M is represented by OD. Since L, M and N are in equilibrium, R and N are in equilibrium.

Therefore R = N and they are opposite in direction.

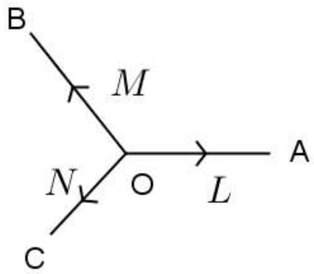
Hence in the triangle OAD, L is represented by OA, M is represented by AD and N is represented by DO

ΔOAD

$$\boxed{\begin{array}{l} L \rightarrow OA \\ M \rightarrow AD \\ N \rightarrow DO \end{array}}$$

3. Lami's Theorem

If three forces acting at a particle are in equilibrium, each force is proportional to the sine of the angle between the other two.



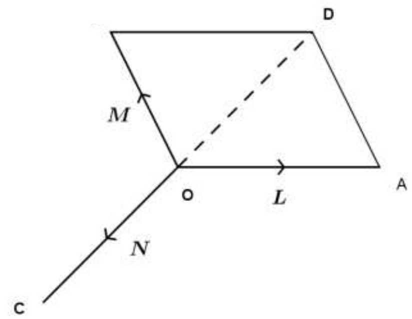
If L, M, N are in equilibrium

$$\frac{L}{\sin\angle BOC} = \frac{M}{\sin\angle COA} = \frac{N}{\sin\angle AOB}$$

This theorem could be easily proved

Using sine rule for a triangle.

From the triangle of forces L, M, N can be represented by the sides of the triangle AOD.In the triangle AOD,

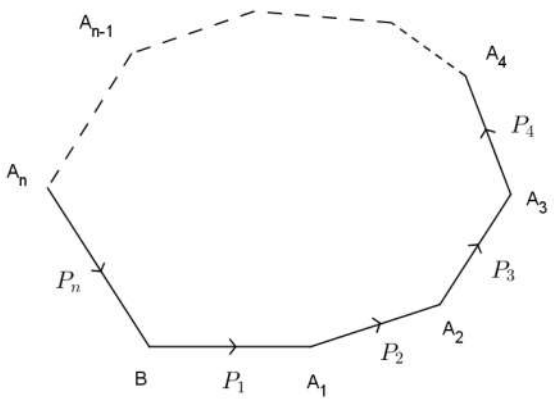


$$\frac{OA}{\sin O\hat{D}A} = \frac{AD}{\sin D\hat{O}A} = \frac{DO}{\sin O\hat{A}D}$$

$$\frac{L}{\sin B\hat{O}C} = \frac{M}{\sin C\hat{O}A} = \frac{N}{\sin A\hat{O}B}$$

4. Polygon of forces

If any number of forces acting on a particle, can be represented in magnitude and direction by the sides of a polygon taken in order, the forces will be in equilibrium.



Let the forces P_1, P_2, P_3, ... P_n act at a particle O and represented by the sides BA_1, A_1A_2,

A_2A_3, ... $A_{n-1}A_n$, A_nB of a polygon $BA_1A_2A_3$ ....A_n

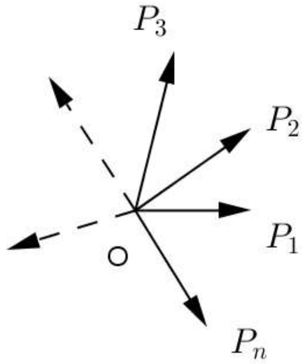
Then the forces are in equilibrium.

$$\overrightarrow{BA_1} + \overrightarrow{A_1A_2} = \overrightarrow{BA_2}$$

$$\overrightarrow{BA_1} + \overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} = \overrightarrow{BA_2} + \overrightarrow{A_2A_3} = \overrightarrow{BA_3}$$

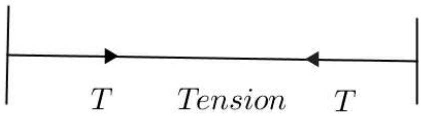
By vector addition,

$$\overrightarrow{BA_1} + \overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \ldots\ldots\ldots\ldots\ \overrightarrow{A_{n-1}A_n} + \overrightarrow{A_nB} = 0$$



Tension of a string

A **ligt string** means that the weight of the string is negligible in comparison with the other weights in the given problem. The force which a string exerts on a body is called the tension and it acts along the string.



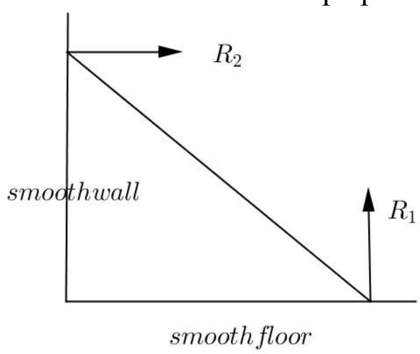
In a light string, the tension in the string is approximately the same throughout its length. If the string is heavy, the tension in the string varies from point to point.

Smooth surfaces

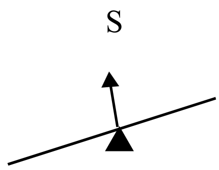
The only force acting between smooth bodies is normal reaction. This normal reaction is perpendicular to their common surface.

i.e , When there is a contact between smooth bodies, the normal reaction is perpendicular to the direction which the body is capable of moving.

The force between a rod and smooth floor is R_1, perpendicular to floor. The force between a rod and smooth wall is R_2, perpendicular to wall. R_1, R_2 are called normal reactions.



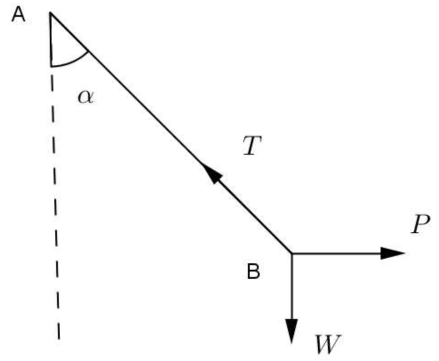
When a rod rests against asmooth peg the reaction S is normal to the rod.



2.7 Worked examples

Example 8

A particle of weight W is attached to one end B of a light string AB and hangs from a fixed point A. A horizontal force P is applied to the particle at B and rests in equilibrium with the string inclined at an angle α to the vertical. Find the tension in the string and the value of P in terms of W and α.



Method I

Forces acting on the particle.

i. Weight W, vertically downwards
ii. Force P, horizontally
iii. Tension T, along the string

For equilibrium of the particle,

Resolving vertically,

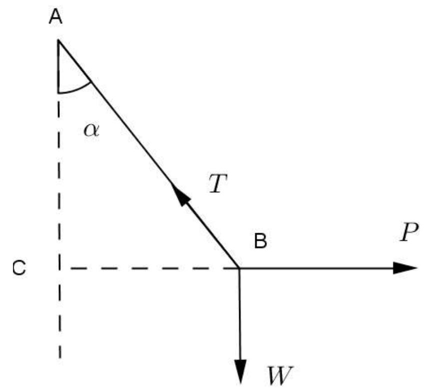
$$\uparrow \quad T\cos\alpha - W = 0 \quad \Rightarrow \quad T = \frac{W}{\cos\alpha}$$

Resolving horizontally,

$$\rightarrow \quad P - T\sin\alpha = 0 \quad \Rightarrow \quad P = T\sin\alpha = W\tan\alpha$$

Method II (Triangle of forces)

Three forces T, W, P act on the particle and the particle is in eqilibrium.
Consider the triangle BAC.



T can be represented by BA; $T \longrightarrow BA$

W can be represented by AC; $W \longrightarrow AC$

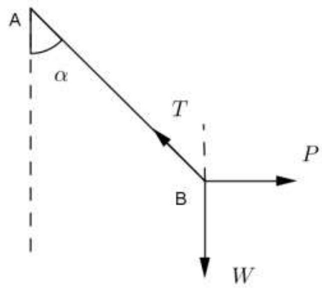
P can be represented by CB; $P \longrightarrow CB$

$$\frac{T}{BA} = \frac{W}{AC} = \frac{P}{CB}$$

$$\frac{T}{BA} = \frac{W}{AC}; \qquad T = W\times\frac{BA}{AC} = \frac{W}{\cos\alpha}$$

$$\frac{W}{AC} = \frac{P}{CB}, \qquad P = W\times\frac{CB}{AC} = W\tan\alpha$$

Method III (Lamis' Theorem)



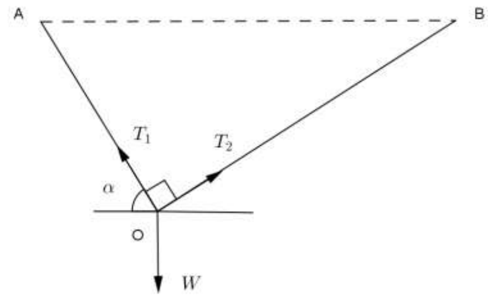
$$\frac{T}{\sin 90} = \frac{W}{\sin(90+\alpha)} = \frac{P}{\sin(180-\alpha)}$$

$$\frac{T}{1} = \frac{W}{\cos\alpha} = \frac{P}{\sin\alpha}$$

$$T = \frac{W}{\cos\alpha}, \qquad P = W\tan\alpha$$

Example 9

A particle of weight W is attached to the ends O of two light strings OA and OB, each of length 50 cm, 120 cm respectively. The other ends A and B are attached to two points at the same level and the distance between A and B is 130 cm. Find the tensions in the string.



$OA^2 + OB^2 = 50^2 + 120^2 = 130^2 = AB^2$

Therefore $\angle AOB = 90°$

If $\angle AOB = \alpha$, $\cos\alpha = \frac{5}{13}$, $\sin\alpha = \frac{12}{13}$

Forces acting at O

i. Weight W, vertically downwards

ii. Tension T_1, along the string OA

iii. Tension T_2, along the string OB

Method I

For equilibrium of the particle.

Resolving horizontally

$$T_2\cos(90 - \alpha) - T_1\cos\alpha = 0$$

$$T_2\sin\alpha - T_1\cos\alpha = 0$$

$$12T_2 - 5T_1 = 0 \quad ....①$$

Resolving vertically,

$$T_2\sin(90 - \alpha) + T_1\sin\alpha - W = 0$$

$$T_2\cos\alpha + T_1\sin\alpha - W = 0$$

$$5T_2 + 12T_1 = 13W \quad ....②$$

From (1) and (2), $T_1 = \frac{12W}{13}$ and $T_2 = \frac{5W}{13}$

Method II (Triangle of forces)

AC is vertical. BO is produced to C.

consider the triangle OAC

$T_1 \longrightarrow OA$

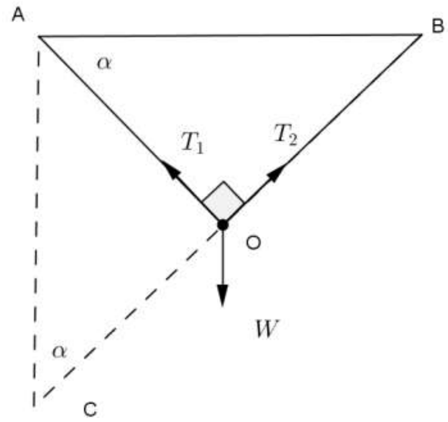
$W \longrightarrow AC$

$T_2 \longrightarrow CO$

$$\frac{T_1}{OA} = \frac{W}{AC} = \frac{T_2}{CO}$$

$$T_1 = W.\frac{OA}{AC} = W\sin\alpha = \frac{12W}{13}$$

$$T_2 = W.\frac{OC}{AC} = W\cos\alpha = \frac{5W}{13}$$



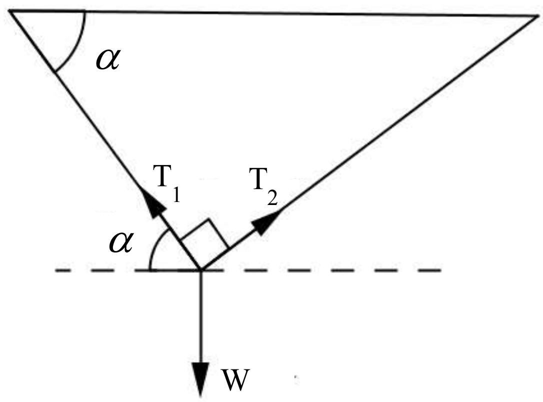
Method III (Lamis Theorem)

$$\frac{T_2}{\sin 90} = \frac{T_1}{\sin(180-\alpha)} = \frac{T_2}{\sin(90+\alpha)}$$

$$\frac{W}{1} = \frac{T_1}{\sin\alpha} = \frac{T_2}{\cos\alpha}$$

$$T_1 = W\sin\alpha = \frac{12W}{13},$$

$$T_2 = W\cos\alpha = \frac{5W}{13}$$



Example 10

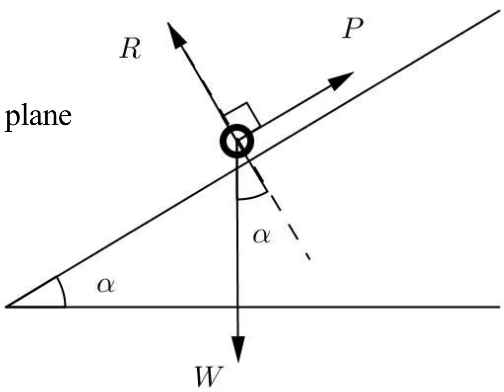
A particle of weight W is placed on a smooth plane inclined at an angle α to the horizontal. Find the magnitude of the force

i. acting along the plane upwards,

ii. acting horizontally

to keep the particle in equilibrium

1. Forces acting on the particle are,
   i. weight W, vertically dowards
   ii. Normal reaction R, perpendicular to the plane
   iii. Force P along the plane



Method I

For equilibrium of the particle,

Resolving along the plane,

$\nearrow$ $P - W\sin\alpha = 0;$ $P = W\sin\alpha$

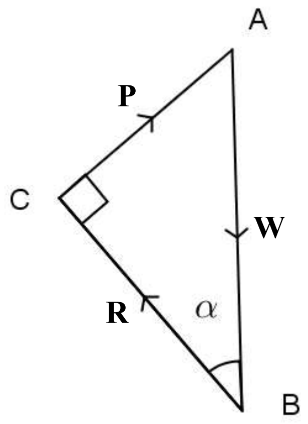
Resolving perpendicular to the plane,

$$\text{M}\ R - W\cos\alpha = 0$$

$$R = W\cos\alpha$$

Method II (Triangle of forces)

Consider the triangle ABC,



$$W \longrightarrow AB$$

$$R \longrightarrow BC$$

$$P \longrightarrow CA$$

$$\frac{W}{AB} = \frac{R}{BC} = \frac{P}{CA}$$

$$R = W.\frac{BC}{AB} = W\cos\alpha$$

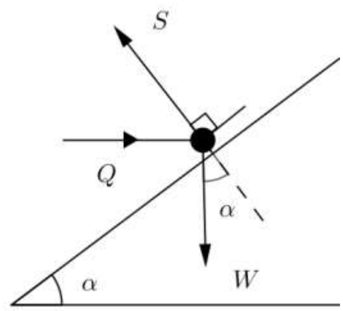
$$P = W.\frac{CA}{AB} = W\sin\alpha$$

(ii). Forces acting on the particle

i. Weight W, vertically downwards

ii. Normal reaction S, perpendicular to the plane

iii. Horizontal force Q



Method I

For equilibrium of the particle

resolving along the plane

$$\text{O}\quad Q\cos\alpha - W\sin\alpha = 0$$

$$\text{M}\quad Q = W\tan\alpha$$

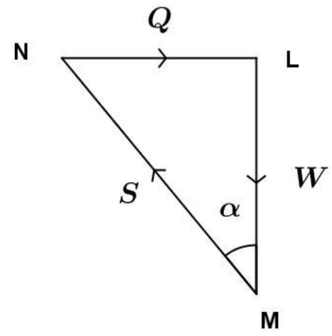
$$S - W\cos\alpha - Q\sin\alpha = 0$$

$$S = W\cos\alpha + Q\sin\alpha$$

$$= W\cos\alpha + \frac{W\sin^2\alpha}{\cos\alpha} = W\left(\frac{\cos^2\alpha + \sin^2\alpha}{\cos\alpha}\right) = W\sec\alpha$$

Method II (Triangle of forces)

Consider the triangle LMN



i) Weight W $\longrightarrow$ LM

ii) Normal reaction S $\longrightarrow$ MN

iii) Horizontal force Q $\longrightarrow$ NL

$$\frac{W}{LM} = \frac{S}{MN} = \frac{Q}{NL}$$

$$Q = W\frac{NL}{LM} = W\tan\alpha$$

$$S = W\frac{MN}{LM} = \frac{W}{\cos\alpha} = W\sec\alpha$$

Example II

A particle of weight W is supported by two strings attached to it. If rhe direction of one string be at α $(0 < \alpha < \frac{\pi}{2})$ to the vertical, find the direction of the other string in order that its tension be minimum. In this case, find the tensions is both strings.

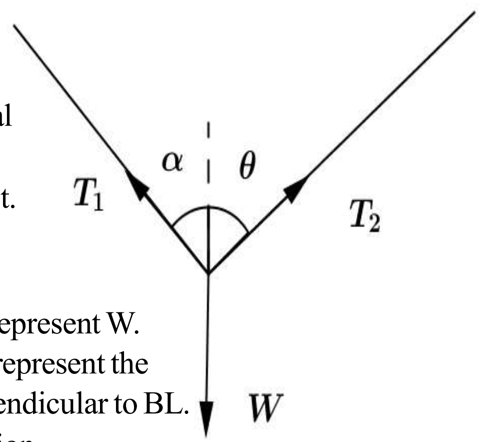
Forces acting on the particle are

i) Weight of the particle W, vertically downwards,

ii) Tension T_1 in the string at an angle α with the vertical

iii)Tension T_2 in the other string T_2 should be minimum

For the equilibrium of the particle, three forces act on it.



This could be done easily using triangle of forces.

Firstly we have to draw AB vertically downwards to represent W.

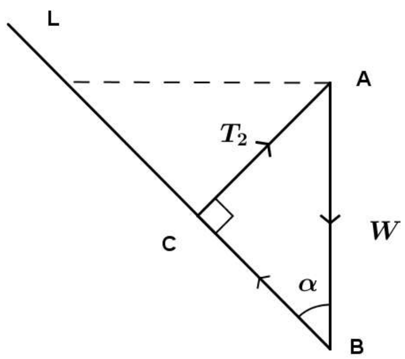
Then draw a line BL at an angle α with the vertical to represent the direction of T_1. For T_2 to be minimum, draw AC perpendicular to BL.

Now T_2 is represented by CA in magnitude and direction.

W $\longrightarrow$ AB

T_1 $\longrightarrow$ BC

T_2 $\longrightarrow$ CA



$$\frac{W}{AB} = \frac{T_1}{BC} = \frac{T_2}{CA}$$

$T_1 = W\cos\alpha$

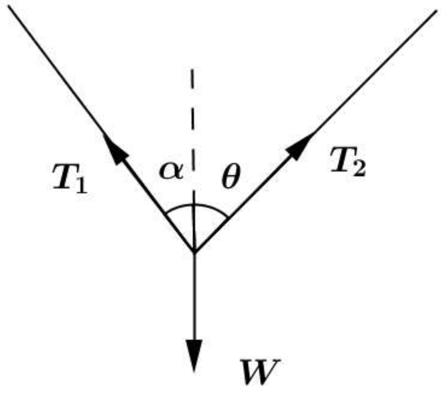
$T_2 = W\sin\alpha$

The direction of the second string (T_2) is perpendicular to the first string.

Method II

For equilibrium of the particle,

applying Lamis theorem



$$\frac{W}{\sin(\alpha+\theta)} = \frac{T_1}{\sin(180-\theta)} = \frac{T_2}{\sin(180-\alpha)}$$

$$T_1 = \frac{W\sin\theta}{\sin(\alpha+\theta)} \qquad T_2 = \frac{W\sin\alpha}{\sin(\alpha+\theta)}$$

For T_2 to be minimum $\sin(\alpha+\theta)$ should be equal to 1. [i.e, $\sin(\alpha+\theta)$ should be maximum]

There fore $\qquad (\alpha+\theta) = \frac{\pi}{2}$

$T_1 = \qquad W\sin\theta = W\sin\left[\frac{\pi}{2}-\alpha\right] = W\cos\alpha$

$T_2 = \qquad W\sin\alpha$

The direction of T_2 is perpendicular to T_1

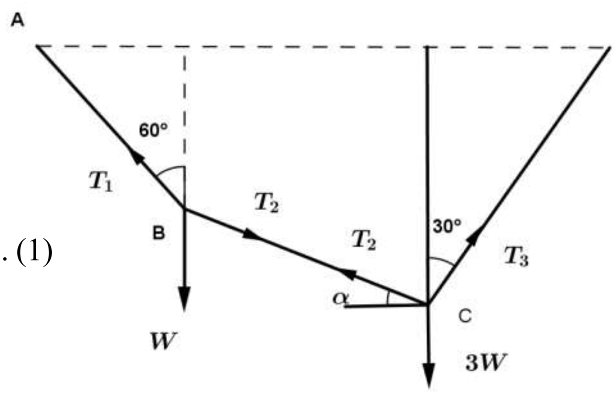
Example 12

The ends A and D of a hight inextensible string ABCD are tied to two fixed points in the same horizonal line. Weights W and 3W are attached to the strings at B and C respectively. AB and CD are inclined to the vertical at angles 60° and 30° respectively. Show that BC is horizontal and find the tensions in the portions AB, BC and CD of the string.

Let BC be at an angle α with the horizontal.

For equilibrium of B,

applying Lamis theorem



$$\frac{T_2}{\sin120} = \frac{T_1}{\sin(90-\alpha)} = \frac{W}{\sin(150+\alpha)}$$

$$\frac{T_2}{\sin60} = \frac{T_1}{\cos\alpha} = \frac{W}{\sin(30-\alpha)} \qquad \text{........ (1)}$$

For equilibrium of C,

$$\frac{T_2}{\sin150} = \frac{T_3}{\sin(90+\alpha)} = \frac{3W}{\sin(120-\alpha)}$$

$$\frac{T_2}{\sin 30} = \frac{T_3}{\cos\alpha} = \frac{3W}{\sin(60+\alpha)} \quad ....(2)$$

From (1) and (2),

$$T_2 = \frac{W\sin 60}{\sin(30-\alpha)} = \frac{3W\sin 30}{\sin(60+\alpha)}$$

$$\sin 60.\sin(60+\alpha) = 3\sin 30.\sin(30-\alpha)$$

$$\frac{\sqrt{3}}{2}\left[\frac{\sqrt{3}}{2}\cos\alpha + \frac{1}{2}\sin\alpha\right] = \frac{3}{2}\left[\frac{1}{2}\cos\alpha - \frac{\sqrt{3}}{2}\sin\alpha\right]$$

$$\sqrt{3}\cos\alpha + \sin\alpha = \sqrt{3}\cos\alpha - 3\sin\alpha$$

$$4\sin\alpha = 0; \quad \sin\alpha = 0; \quad \alpha = 0$$

Hence BC is horizontal

From (1) $\quad T_1 = \frac{W}{\sin 30} = 2W$

Form (1) $\quad T_2 = \frac{W\sin 60}{\sin 30} = \sqrt{3}\,W$

Form (2) $\quad T_3 = \frac{3W}{\sin 60} = 2\sqrt{3}$

Example 13

(a) Fores $F_1 = 4\underline{i} + 2\underline{j}$, $F_2 = 2\underline{i} - 5\underline{j}$ and $F_3 = -\underline{i} + \underline{j}$ act at a point. Find the magnitute and direction of the resultant of three forces.

(b) The coordinates of three points A, B and C are A(2,3), B(5,7) and C(-3,15)

i. Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in terms of $\underline{i}, \underline{j}$

ii. Fores $\underline{F}_1$ and $\underline{F}_2$ of magnitudes 20 N and 65 N respectively act at the point A, along AB and AC respectively. Find the magnitude and direction of the resultant.

(unit vectors along the coordinate axes Ox and Oy are $\underline{i}$ and $\underline{j}$ respectively.)

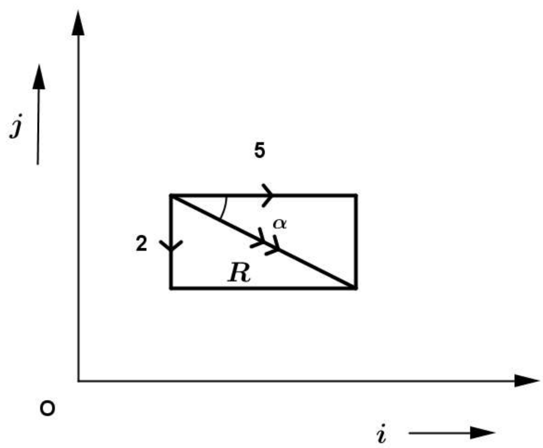
(a) $\underline{R} = \underline{F}_1 + \underline{F}_2 + \underline{F}_3$

$$= (4\underline{i} + 2\underline{j}) + (2\underline{i} - 5\underline{j}) + (-\underline{i} + \underline{j})$$

$$= 5\underline{i} - 2\underline{j}$$

$$|\underline{R}| = \sqrt{5^2 + 2^2} = \sqrt{29}$$

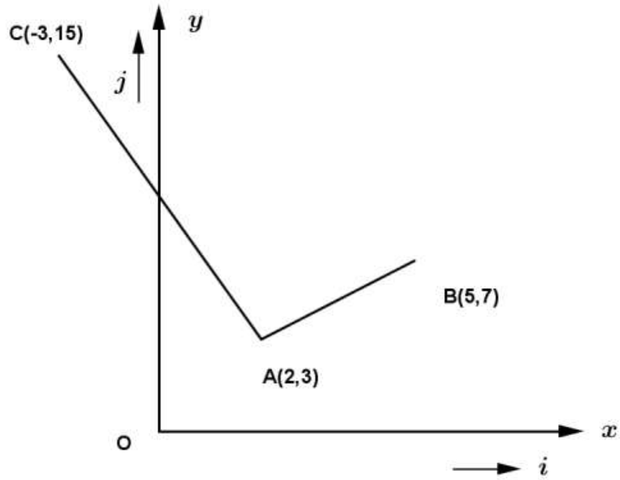
$$\tan\alpha = \frac{2}{5}, \; \alpha = \tan^{-1}\left(\frac{2}{5}\right)$$



b) A $\equiv$ (2, 3), B $\equiv$ (5, 7), C $\equiv$ (-3, 15)

$\overrightarrow{OA} = 2\underline{i} + 3\underline{j}$, $\overrightarrow{OB} = 5\underline{i} + 7\underline{j}$,

$\overrightarrow{OC} = -3\underline{i} + 15\underline{j}$



$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (5\underline{i} + 7\underline{j}) - (2\underline{i} + 3\underline{j})$

$= 3\underline{i} + 4\underline{j}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= (-3\underline{i} + 15\underline{j}) - (2\underline{i} + 3\underline{j})$

$= 5\underline{i} + 12\underline{j}$

Unit vector along $\overrightarrow{AB}$ is $\frac{1}{5}(3\underline{\boldsymbol{i}} + 4\underline{\boldsymbol{j}})$

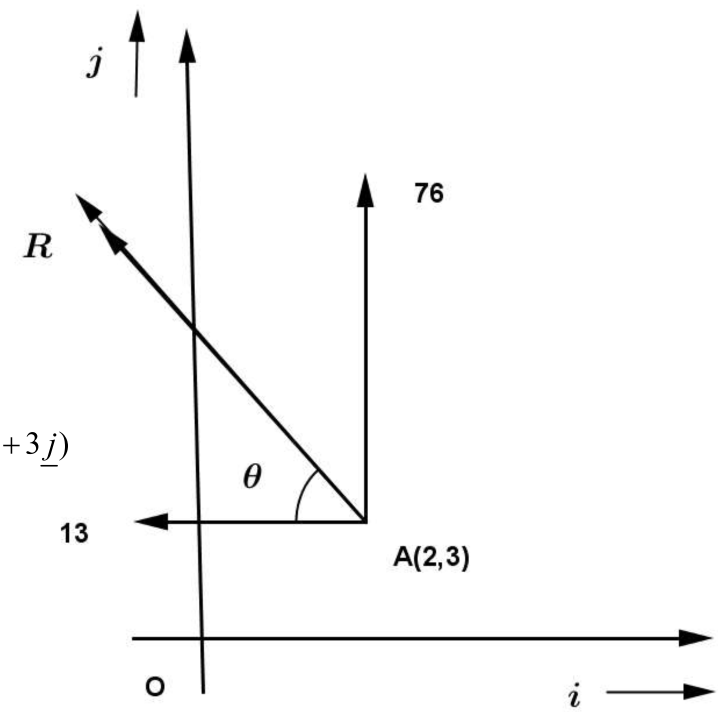
Unit vector along $\overrightarrow{AC}$ is $\frac{1}{13}(-5\underline{\boldsymbol{i}} + 12\underline{\boldsymbol{j}})$

$F_1 = 20 \times \frac{1}{5}\ (3\underline{i} + 4\underline{j})$

$= 12\underline{i} + 16\underline{j}$

$F_2 = 65 \times \frac{1}{13}\ (-5\underline{i} + 12\underline{j})$

$= -25\underline{i} + 60\underline{j}$



Resultant $\underline{R} = \underline{F}_1 + \underline{F}_2$

$= (12\underline{i} + 16\underline{j}) + (-25\underline{i} + 3\underline{j})$

$= -13\underline{i} + 76\underline{j}$

$|\underline{R}| = \sqrt{13^2 + 76^2}$

$\theta = \tan^{-1}\left(\frac{76}{13}\right)$

2.8 Exercises

1. Two forces P and Q act on a point at an angle θ. The resultant is R and α is the angle between R and P.

   a) $P = 6, Q = 8, \theta = 90°$; find R and α

   b) $P = 10, Q = 8, \theta = 60°$; find R and α

   c) $P = 15, Q = 15\sqrt{2}, \theta = 135°$; find R and α

   d) $P = 8, R = 7, \theta = 120°$; find Q and α

   e) $P = 7, R = 15, \theta = 60°$; find Q and α

2. The forces F and 2F act on a particle. The resultant is perpendicular to F. Find the angle between the forces.

3. The forces P and 2P. Newton act on a particle. If the first be doubled and second be increased by 10 newtons, the direction of the new resultant is unchanged. Find the value of P.

4. Two forces P and Q act on a particle at an angle θ. When θ is 60°, the resultant is $\sqrt{57}$ N and when θ is 90° the resultant is $5\sqrt{2}$ N. Find P and Q.

5. If the resultant of two equal forces inclined at angle 2θ is twice the magnitude of the resultant when they at an angle 2α, show that $\cos\theta = 2\cos\alpha$.

6. Two forces P and Q act at an angle θ. The resultant is equal to P in magnitude. When P is doubled the new resultant also equals to P in magnitude. Find Q in terms of P and the value of θ.

7. The forces P, P, $\sqrt{3}$ P act on a particle and keep it in equilibrium. Find the angle between them.

8. The resultant of two forces P and Q is $\sqrt{3}$ Q and makes an angle 30° with P. Show that $P = Q$ or $P = 2Q$.

9. ABCD is a square. Forces P, $2\sqrt{2}$ P, 2P act at A along AB, AC, AD respectively. Find the resultant.

10. ABCD is a rectangle. AB = 3m, BC = 5 m. Forces 6, 10, 12 newton act at A along AB, AC, AD respectively. Find the resultant.

11. ABCDEF is a regular hexagon. Forces $2\sqrt{3}$, 4, $8\sqrt{3}$, 2 and $\sqrt{3}$ newton act at B along BC, BD, EB, BF and AB respectively in the directions indicated by the order of the letters. Find the resultant.

12. ABCD is a square. E and F are the midpoints of BC and CD respectively. Forces 5, $2\sqrt{5}$, $5\sqrt{2}$, $4\sqrt{5}$, 1 newton act at A along AB, AE, CA, AF, AB respectively in the directions indicated by the order of the letters. Find the resultant.

13. ABCD is a square of 4 cm. The point E, F, G, H and J lie on the sides AB, BC, CD, DA respectively such that AE = BF = CG = HD = DJ = 1 cm. (Note that G and H lie on CD and CG = 1 cm, GH = 2 cm). Forces of magnitudes 10, $3\sqrt{10}$, $2\sqrt{5}$, 10, $\sqrt{10}$, 5 newton act at the point E in the directions EB, EF, EG, EH, EJ, EA respectively. Find the resultant of these forces.

14. ABC is an equilateral triangle and G is its centroid. Forces 10, 10 and 20 newton act at G along GA, GB and GC respectively. Find the nagnitude and direction of their resultant.

15. A particle of weight 50 N is suspended by two light string of lengths 60 cm and 80 cm from two point at the same level and 100 cm apart. Find the tensions in the strings.

16. A particle of weight 100 N is placed on a smooth plane inclined at 60 to the horizontal. What force applied

    (a) parallel to the surface of the plane

    (b) horizontally

    will keep the particle at rest?

17. A particle of weight 30 N is suspended from two points A, B 60 cm apart and in the same horizontal line, by the strings of length 35 cm and 50 cm. Find the tension in each string.

18. A string of length 120 cm is attached to two points A and B at the same level at a distance of 60 cm apart. A ring of 50 N can slide freely along the string, is acted on by a horizonal force. F which holds it in equilibrium vertically below B. Find the tension in the string and the magnitude of F.

19. A string is tied to two points at the same level, and a smooth ring of weight W N can slide feely along the string is pulled by a horizontal force F N. In the equilibrium position, the portions of the strings at an angle 60° and 30° to the vertical. Find the value of F and tensions in the strings.

20. Ox, Oy are perpendicular axes and the unit vectors in the directions of Ox and Oy are $\underline{i}$ and $\underline{j}$ respectively.

a) Forces $\underline{F}_1 = 3\underline{i} + 5\underline{j}$, $\underline{F}_2 = -2\underline{i} + \underline{j}$, $\underline{F}_3 = 3\underline{i} - \underline{j}$ act on a particle. Find the magnitude and direction of the resultant of $\underline{\mathbf{F}}_1$, $\underline{\mathbf{F}}_2$ and $\underline{\mathbf{F}}_3$.

b) Forces $R_1 = (2P\underline{i} - P\underline{j})$, $R_2 = (-4\underline{i} + 3P\underline{j})$ and $R_4 = (2Q\underline{i} - 5\underline{j})$ act on a particle and it is in equilibrium. Find the values of P and Q.

c) The coordinate of two points A and B are (3, 4) and (-1, 1) respectively. 2, 3, 5, $6\sqrt{2}$ newtons act at O, along Ox, Oy, OA, OB respectively. Express each force in the form $X\underline{i} + Y\underline{j}$ and hence calculate the magnitude and direction of the resultant of the four forces.

21. The unit vectors along rectangular cartesion axes Ox, Oy are $\underline{i}$, $\underline{j}$ respectively. Two forces P and Q acting on a particle are parallel to the vectors $4\underline{i} + 3\underline{j}$ and $-3\underline{i} - 4\underline{j}$ respectively. The resultant of the two forces is a force of magnitude 7N acting in the direction of vector $\underline{i}$. Calculate the magnitudes of P and Q.

3.0 Parallel Forces, Moments, Couples

3.1 Parallel Forces

In chapter two we have shown how to find the resultant of forces which act at a point. Now in this chapter we shall consider the action of parallel forces and the way to find their resultant.

Two types of parallel forces:

i. Like parallel forces

Two parallel forces are said to be like parallel forces when they act in the same direction (sense)

ii. Unlike parallel forces

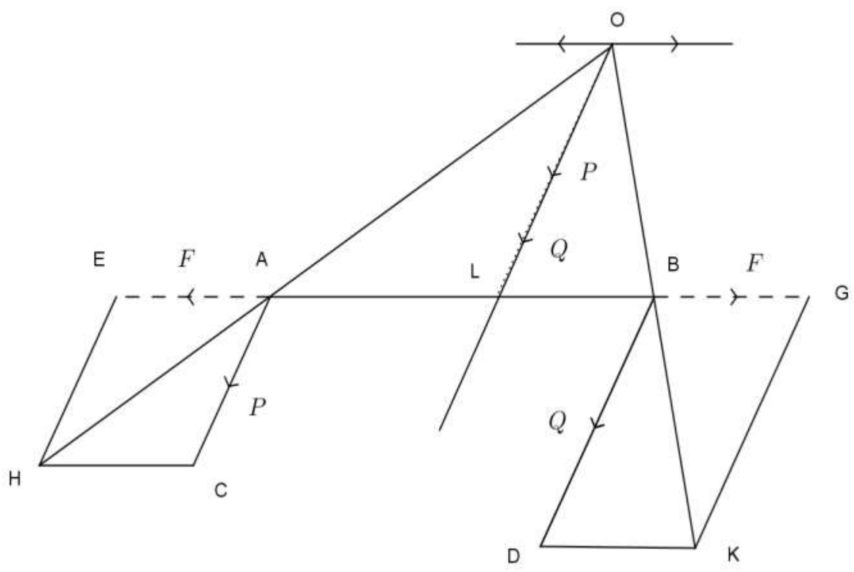
When two parallel forces act in the opposite parallel direction, they are said to be unlike.

Since parallel forces do not meet at a point their resultant cannot be obtained by direct application of paralleogram forces.

Resultant of two like parallel forces

Consider two like parallel forces P and Q acting at points A and B represented by lines AC and BD respectively.

At A and B introduce two equal and opposite forces F acting along the line AB as ahown represented by AE and BG. These equal and opposite forces balance each other and have no effect on P and Q.



Complete the parallelograms AEHC and BDKG and produce the diagonals HA, KB to meet at O.

Draw OL parallel to AC (or BD) to meet AB at L.

The resultant of P and F at A, represented by AH and the resultant of Q and F at B, represented by BK may be supposed to act at O along OAH and OBK respectively.

These resultant forces may be resolved at O. The components are P along OL, F parallel to AE and Q along OL and F parallel to BG. Equal and opposite forces F at O balance each other. Hence the resultant of original forces P and Q is a forces (P + Q) parallel to original direction along OL.

Finding the position of L. The triangles OLA, ACH are simillar.

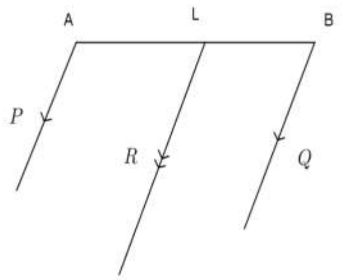
$$\frac{OL}{LA} = \frac{AC}{CH} = \frac{P}{F} \quad \text{...} ①$$

and also the triangles OLB, BDK are similar.

$$\frac{OL}{LB} = \frac{BD}{DR} = \frac{Q}{F} \quad \text{...} ②$$

From ① and ② , OL × F = P × LA = Q × LB

$$\frac{LA}{LB} = \frac{Q}{P}$$



ie. The point L divides AB internally in the ratio of the forces.

P . AL = Q.BL and the resultant R = P + Q

Note : When P = Q , resultant R bisects AB.

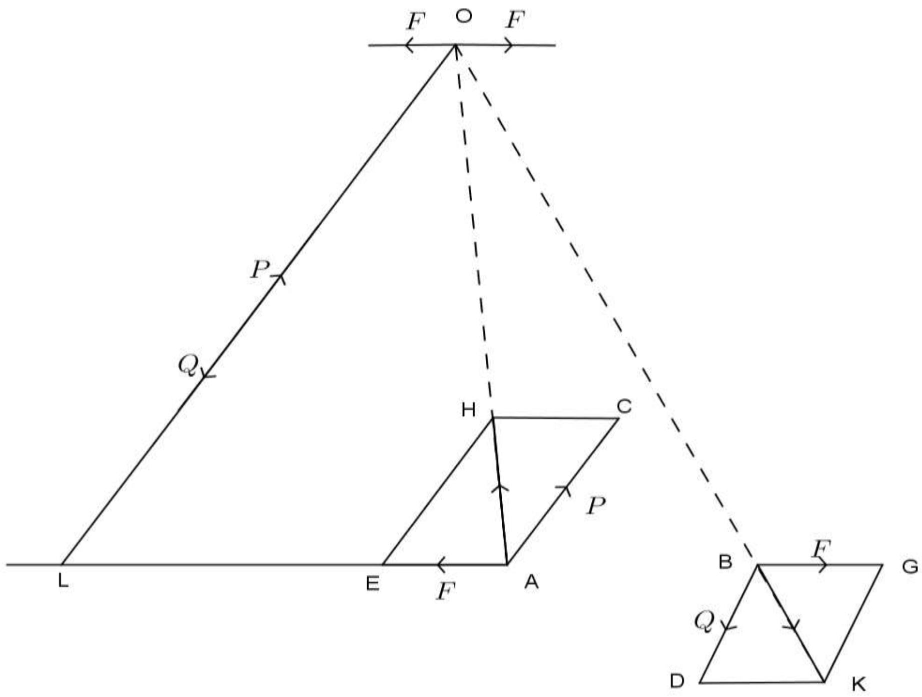
Case (ii)

Resultant of two unlike forces

Consider two unlike parallel forces P and Q (P > Q) acting at points A and B represented by AC and BD respectively.

At A and B introduce two equal and opposite forces F, acting along the line AB represented by AE and BG. They balance each other and have no effect on P and Q. Complete the parallelograms AEHC, BGKD and produce the diagond AH, KB to meet at O. (They always meet at a point unless they are equal in magnitude, P = Q).

Draw OL parallel to CA (or BD) to meet BA produced at L.



The resultant of forces P and F at A, represented by AH and the resultant of forces Q and F at B, represented by BK may be supposed to act at O along AO and OB espectively. These resultant forces may be resolved at O. The components are P along LO, F parallel to AE and Q along OL, F parallel to BG. Equal and opposite forces at F balance each other. Hence the resultant of original forces P and Q is a single force (P - Q) acting along LO parallel to P in the direction of P.

The position of point L

by construction triangles OLA and HEA are similar

$$\frac{OL}{LA} = \frac{HE}{EA} = \frac{P}{F} \qquad \text{.....................................} \ ①$$

and also the triangles OLB, BDK are similar.

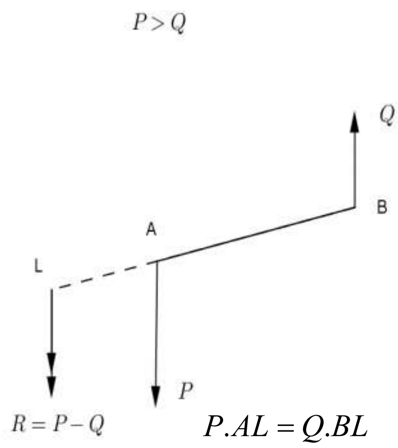
$$\frac{OL}{LB} = \frac{Q}{F} \quad = \qquad \text{.....................................} \ ②$$

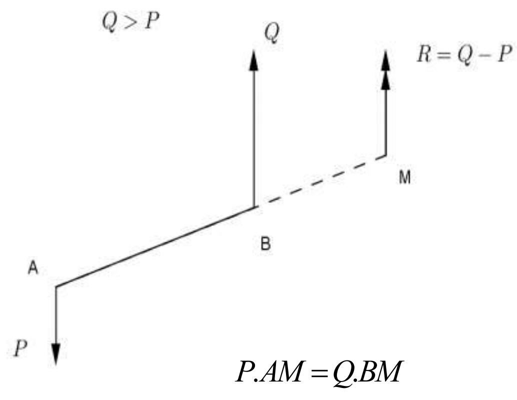
From ① and ② $\qquad \frac{LA}{LB} = \frac{Q}{P}$

ie The point L divides AB externally in the inverse ratio of the forces

Note:

When P = Q, the triangles AEH and BGK are concruent so that diagonals AH and KB being parallel and will not meet at point O. Hence the construction fails, lead to the conclution no single force is equivalent to two equal unlike parallel forces. Such a pair of forces consitutes a couple





will be discussed latter.

To find the resultant of any number of parallel forces.

(i) If the forces are like parallel.

The resultant force can be obtained by the repeated application of finding the resultant of two like forces till all the forces have been taken.

The resultant will be the sum of all the forces and its direction is same as the direction of given forces if the forces are unlike.

(i) If the forces are unlike paralls.

Divide the forces into two sets of like forces and find their resultant forces as mentioned above. Then find the resultant of a pair of unlike parallel forces as given below.

a) If they are unequal, the resultant force is a single force with algebraic sum of the given forces as its magnitute.

b) (i) If they are equal and the line of action are coincident, no resultant force and all the given forces are in equilibrium.

(ii) If they are equal and line of action are not coincident they form a couple.

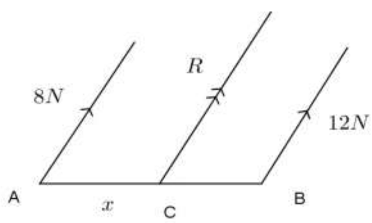
3.2 Worked examples

Example 1

1) Like parallel forces of 8 and 12 N act at points A and B where AB = 15 cm

a. Find the magnitude of resultant and the point where the resultant cuts AB.

b. When these forces are unlike find the resultant and the position of the line of action.



(a) $R = P+Q = 8+12 = 20N$

$8.AC = 12.BC$

$8x = 12(15-x)$

$20\,x = 12 \times 15$

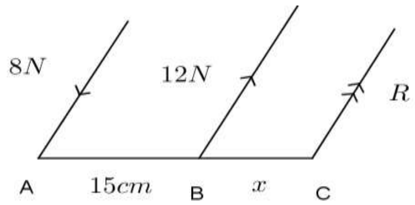
$AB = 9$ cm

(b) $R = 12-8 = 4N$

$12x = (15+x)8$

$4x = 15 \times 8$

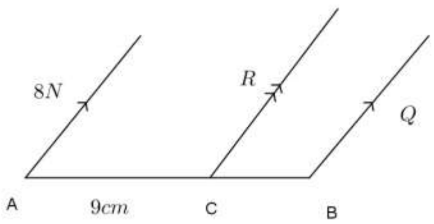
$x = 30$ cm



2) In the following examples A and B are the points where parallel forces P, Q acts and C be the point that the resultant R meets AB.

i. P and Q are like parallel forces, P = 8 N, R = 17 N, AC = 9 cm
find Q and AB

ii. P, Q are unlike forces P = 6N AC = 18 cm , CB = 16 cm
find Q and R



$P + Q = 17$

$Q = 17 - 8$

$= 9$ N

$AC:CB = 9:8$

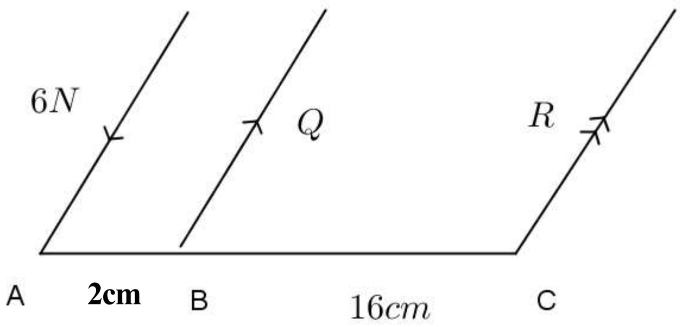
$6 \times 18 = Q \times 16$

$Q = \frac{27}{4}$

$R = Q-P$

$R = \frac{27}{4} - 6$

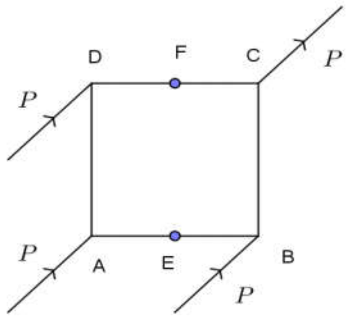
$R = \frac{3}{4}$



3) Four equal like parallel forces act at the verices of a square show that the resultant passes through the centre of the square

Let the forces are PN.

Resultant of P at A and P at B is a like parallel force 2P at E when E is the midpoint of AB.



and resultant of P at C and P at D is also a like parallel force of magnitute 2P N acting at F. where F is the mid-point of CD.

Now resultant of two like parallel forces of 2P and 4P acting through the midpoint of EF which coincides with the centre of the square

Therefore resultant passes through the centre of the square.

4) P and Q are like parallel forces. If Q in moved parallel to itself through a distance ***x*** prove that the resultant of P and Q moves through a distance

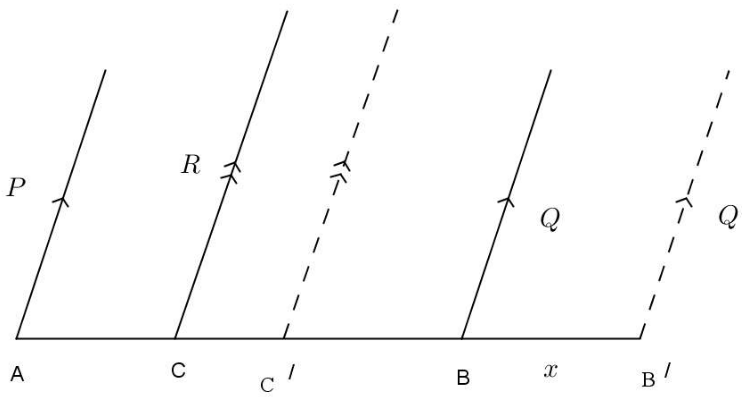
Let R be the resultant of forces P and Q acting at A and B and pass through the point C in AB

Then

$$\frac{AC}{CB} = \frac{Q}{P}$$

$$\frac{AC}{AB} = \frac{Q}{P+Q}$$

$$AC = \left(\frac{Q}{P+Q}\right)AB$$



Now Q moved a distance x then the resultant act at $C^{/}$ is AB

then

$$\frac{AC^{/}}{C^{/}B^{/}} = \frac{Q}{P}$$

$$AC^{/} = \left(\frac{Q}{P+Q}\right)AB^{/} = \left(\frac{Q}{P+Q}\right)(AB + x)$$

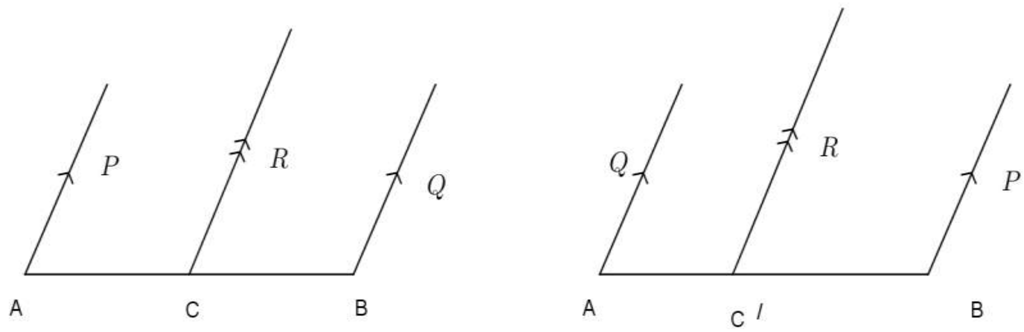
Distance moved by the resultant

$$CC^{/} = AC^{/} - AC$$

$$CC^{/} = \left(\frac{Q}{P+Q}\right)[AC + x - AC]$$

$$CC = \left(\frac{Q}{P+Q}\right)x$$

5) Two like parallel forces P and Q act on a rigidbody at A and B respectively. If P and Q interchanged show that the point of the resultant cuts AB will move through a distance.AB will move through a distance $\left(\frac{P-Q}{P+Q}\right)AB$



$$\frac{AC}{CB} = \frac{Q}{P}$$

$$AC = \left(\frac{Q}{P+Q}\right)AB$$

$$\frac{AC^{/}}{C^{/}B} = \frac{P}{Q}$$

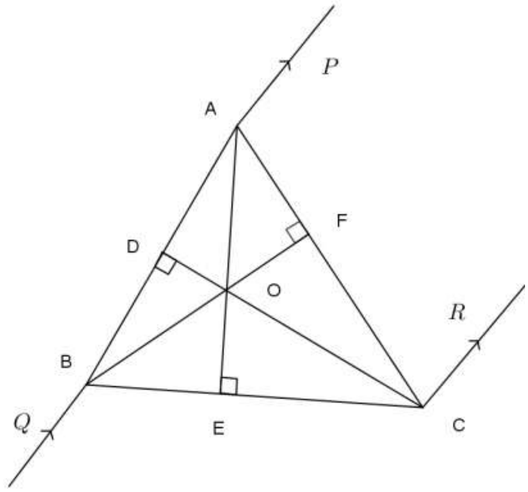
$$AC^{/} = \left(\frac{P}{P+Q}\right)AB$$

$$AC^{/} - AC = \left(\frac{P}{P+Q}\right)AB - \left(\frac{Q}{P+Q}\right)AB$$

$$= \left(\frac{P-Q}{P+Q}\right)AB$$

6) Like parallel forces P, Q, R act at the verhicles of a triangle ABC. Show that if the resultant passes through the orthocentre of the triangle.

P : Q : R = tan A : tan B : tan C



O be the orthocentre of the triangle

Given that the resultant passes through O.

Resultant of P and Q should pass through D, where **CD $\perp$ AB**

$$\frac{AD}{DB} = \frac{Q}{P} = \frac{CD \cot A}{CD \cot B}$$

$$\frac{Q}{P} = \frac{\tan B}{\tan A} \quad \text{...(1)}$$

similarly resultant of Q and R should pass through E. Where AE $\perp$ BC

$$\frac{BE}{EC} = \frac{R}{Q} = \frac{AE \cot B}{AE \cot C} = \frac{\tan C}{\tan B} \quad \text{..................................(2)}$$

(1) , (2) $\Rightarrow$ $P : Q : R = \tan A : \tan B : \tan C$

3.3 Exercises

1. Like parallel forces of magnitude 2, 5, 3 N act at the vertices A, B, C of a triangle ABC respectively. Where AB = 4 cm, BC = 3 cm and AC = 5 cm

   Find i) Magnitude of the resultant

   ii) The position of the line action of the resultant

2. Like parallel forces of magnitude P, P, 2P act at the verticles A, B, C of a triangle ABC. Show that the resultant passes through the midpoint of the line joining C to the midpoint of AB.

3. four equal like parallel forces act at the verticles of a square show that their resultant passes the centre of the square.

4. Three like parallel forces P, Q, R act at the verticles ABC of a triangle ABC. If the resultant passes through the incentre of the triangle prove that

$$\frac{P}{BC}=\frac{Q}{AC}=\frac{R}{AB}$$

5. Four forces are represented by $\overline{AB}, 2\overline{BC}, 3\overline{CD}$ and $4\overline{DA}$. Where ABCD is a square. Show that their resultant is represented in magnitude and direction by .

6. Two unlike parallel forces P and Q (P > Q) act at A and B respectionly. If P and Q are increasted by S. Show that the resultant will move by a distance . $\dfrac{S.AB}{P-Q}$

7. Three like parallel forces P, Q and R act at the verticles A, B, C of a triangle ABC. If the resultant passes through

   (i) The centroid show that P = Q = R

   (ii) The circumcentre show that $\dfrac{P}{\sin 2A}=\dfrac{Q}{\sin 2B}=\dfrac{R}{\sin 2C}$

8. Three parallel forces of magnitudes P, 2P, 3P act through the points A, B, C respectively on a stranght line OABC where OA = a, AB = b and BC = c. Show that the resultant act through the point D in OABC where OD = $\dfrac{6a+5b+3c}{2}$

3.4 Moments

Forces acting on a rigidbody may tend to rotate the body, if one point of the body is fixed. The tendency of the force to turn the body introduces the idea of moment of a force about a point.

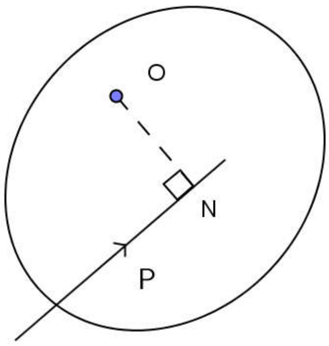
If a single force acts on a rigidbody which one point is fixed the force will tend to turn the body if the line of action of the force does not pass through that point.

Def :

The moment of a force about a given point is the product of the force and the perpendicular distance from the point to the line of action of the force.

Note :

When the line of action passes through the point O the moment about that point O is zero.



O is a fixed point on the body ON is the perpendicular drawn from O to the line of action of the force P. Then the moment of force P about O is P × ON and it tends to turn the body in the anticlockwise sense.

moment about O is = P × ON m

The SI unit of moment is Newton metre, Nm, moments are positive or negative accroding the tend of anticlockwise or clockwise rotation about the point.

When a certain number of forces acting on a body the algebraic sum of their moments is obtained by adding the moments of each force about that point with its sign.

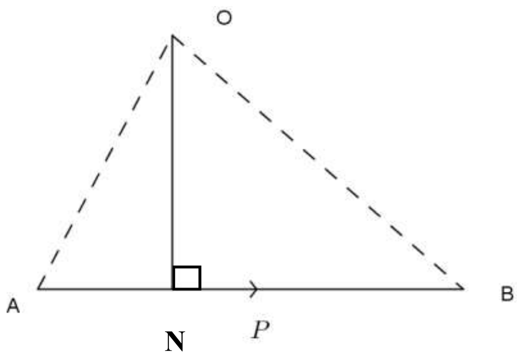
The moment of a force is a vector quantity as it has magnitude and direction (sense)

Graptical representation of moment

Suppose the force P is represented by line segment AB in magnitude and direction. Let O be a point about which moment to be taken. ON is the perpendicular from O to AB, then the moment of force P about O is P × ON m = AB × ON

But area of triangle OAB = $\frac{1}{2}$ AB × BN

Hence twice the area of the triangle AOB, whose base represents the force and vertex is the point about which moment to be taken is numerically equal to the moment of the force about that point. Hence P. ON = 2ΔOAB



Note:

The graphical representation is used to prove some fundamental theorems about moment.

Varignon's Theorem

The algebraic sum of moments of any two coplaner forces about any point in their plane is equal to the moment of their resultant about the same point. We have two cases to consider

(i) forces are non parallel

(ii) forces are parallel

case (i) When the forus are non parallel.

Proof : When forces meet at a point. Let P and Q be the forces acting at A and O be the point in their plane and the moment is to be taken about O. Draw OC parallel to the direction of P to meet the line of action of Q at C.

Let AC represents Q in magnitude and on the same scale AB to represent force P.

Complete the parallelogram ABCD. Join OA and OB. AD represents the resultant R of P and Q.

O have two passibilities as shown above

In both, moment of P about O is m 2ΔOAB

moment of Q about O is m 2ΔOAD

and moment of R about O is 2ΔOAC

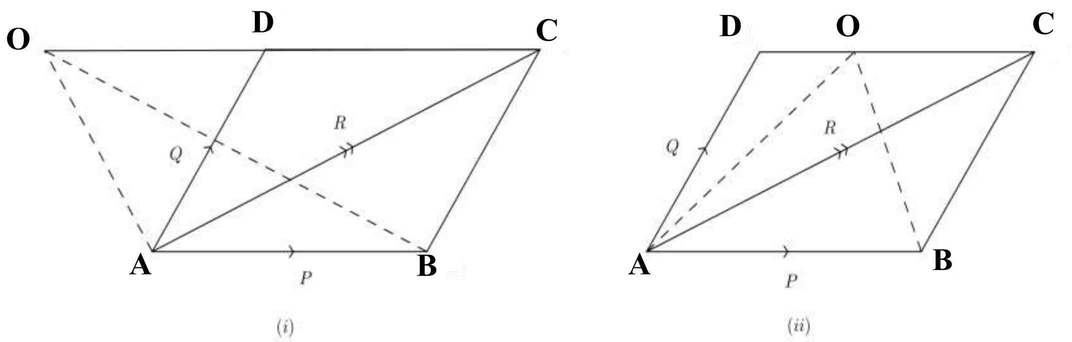
In figure (i) sum of the moments of P and Q is = 2ΔOAB + 2ΔOAD m

= 2ΔABC + 2ΔOAD

= 2ΔACD + 2ΔOAD m

= 2ΔOAC m

= moment of R about Om



In figure (ii)

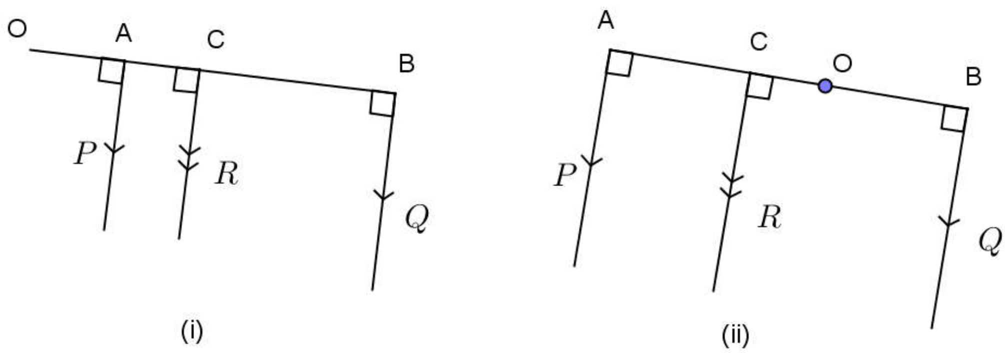
sum of themoments of P and Q is $= \Delta OAB - 2\Delta AOD$ m

$= \quad 2\Delta ADC - 2\Delta AOD$ m

$= \quad 2\Delta AOC$ m

= moment of R about O

case (ii) When the forces are parallel



Let P and Q be two like parallel forces acting and O be any point in their plane as shown above

Draw OAB perpendicular to the forces to meet their lines of action at A and B.

Let R be the resultant of P and Q and acts through C, where OC is perpendicular to R and AC : CB = Q : P

In figure (i) sum of the moments of P and Q about O = $P \times OA + Q \times OB$ ○

$= P(OC - AC) + Q(OC + CB)$ ○

$= (P + Q)OC - P \times AC + Q \times CB$ ○

Since $\dfrac{AC}{CB} = \dfrac{Q}{P}$

$$P \times AC = Q \times CB$$

sum of the moments = (P + Q) × OC ○

= moment of R about O.

In figure (ii) sum of the moments of P and Q is = P × OA m + Q × OB○

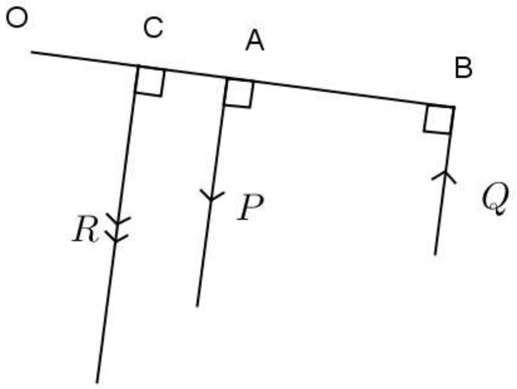
= P × OA - Q × OB m

= P(OC + CA) - Q(CB - OC)

= (P + Q)OC + P × AC - Q × CB

= (P + Q) OC m

= moment of R about O.



When forces are unlike and parallel

Let P, Q be unlike parallel forces and P > Q

then R= P - Q

sum of the moment about O

= P × OA - Q × OB

= P(OC + CA) - Q(OC + CB)

=(P - Q)OC + P × AC - Q × CB

= (P - Q) OC ○

= moment of R about O.

Note : The algebraic sum of moments about any point in the line of action of their resultant is zero.

Generalised Theorem

This is known as principle of moments. If any number of coplaner forces acting on a rigidbody has a resultant, the algebraic sum of moments of the forces about any point in their plane is equal to the moment of their resultant about that point.

If a system of coplaner forces is in equilibirium their resultant is zero and its moment about any point in their plane must be zero.

If a system of coplaner forces is in equilibirium then the algebraic sum of their moment about any point in their plane is zero.

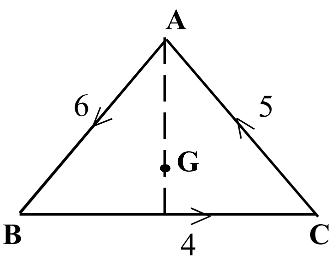
The convence, is not true.

If the sum of moments of a system of a coplanar forces about one point in their plane is zero does not mean the system of forces is in equilibrium, for the point may lie on the line of action of the result.

3.5 Worked examples

Example 7

Forces of 4, 5, 6N act along the sides BC, CA and AB of an equilateral triangle ABC of side 2 m in the direction indicated by the order of letters. Find the sum of their moments about the centroid of the triangle.



Let G be the centroid $\quad AD = 2\sin 60$

$$= 2\frac{\sqrt{3}}{2} = \sqrt{3}m$$

$$GD = GE = GF = \frac{1}{3}\sqrt{3}m$$

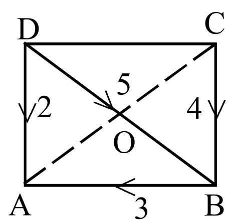
$$\text{Sum of moments abou G} = 4\times\frac{1}{\sqrt{3}} + 5\times\frac{1}{\sqrt{3}} + 6\times\frac{1}{\sqrt{3}}$$

$$= \frac{15}{\sqrt{3}}\text{m} = 5\sqrt{3}\,\text{Nm m}$$

Example 8

The side of a square ABCD is 24m. Forces of 4, 3, 2 and 5N act along CB, BA, DA and DB respectively as indicated by the order of lettes. Find the sum of their moments about

(i) vertex C (ii) The centre of the square O



$$CO = 4\cos 45 = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

$$\text{Sum of moments about Cm} = 2\times 4 - 3\times 4 + 5\times 2\sqrt{2}\ \text{m}$$

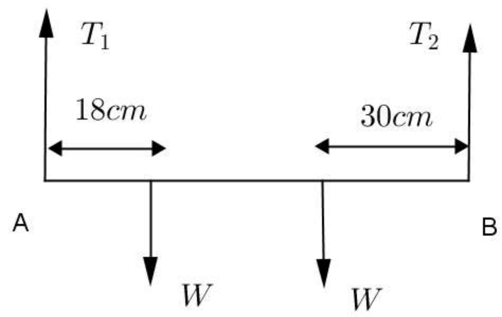
$$= (10\sqrt{2} - 4)\ \text{Nm}$$

$$\text{Sum of moments about O} \circ = 4\times 2 + 3\times 2 - 2\times 2 \circ$$

$$= 10\ \text{Nm} \circ$$

Example 9

A light rof of 72 cm has equal weights attached to it, one at 18 cm from one end and other at 30 cm from other end. The rod is suspended in a horizontal position by two vertical strings attached to the ends of the rod. If the strings can just support a tension of 50 N find the magnitude of the greatest weight that can be placed.



Let the equal weight be W and the tension in the strings be T_1, T_2N.

for equilibrium of the rod $\uparrow$ $T_1 + T_2 - 2W = 0$

Bm $\quad -T_1 \times 72 + W \times 54 + W \times 30 = 0$

$72T_1 = 84W$

When T_1 is maximum ($T_1 = 50$)

$72 \times 50 = 84W$

$$W = \frac{72 \times 50}{84} = 42\frac{6}{7}N$$

Am $\quad T_2 \times 72 - W \times 18 - W \times 42 = 0$

$72T_2 = 60$ W, When T_2 ismaximum ($T_2 = 50$)

$$W = \frac{72 \times 50}{60} = 60 \text{ N}$$

Therefore the greatest weight can be placed is $42\frac{6}{7}$N

Example 10

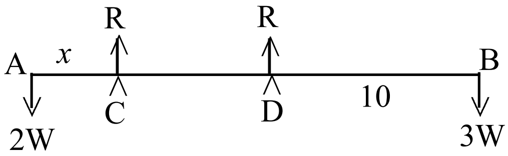
A light rod of AB 20cm long rests on two pegs whose distance apart is 10 cm. Weights of 2W and 3W are suspended from A and B. Find the position of the pegs so that the reaction of the pegs be equal.

Let the distance of one from A is x cm.

The rod is in equilibirium. Therefore sum of moments taking moments about C, is zero.

$$R \times 10 + 2Wx - 3W(20 - x) = 0$$

$$10R = 60W - 5Wx \quad \text{..............................} (1)$$



taking moments about O

$$R \times 10 + 3W(10 - x) - 2W(10 + x) = 0$$

$$10R = 5Wx - 10W \quad \text{.................................(2)}$$

(1) & (2) $\quad 10x = 70$

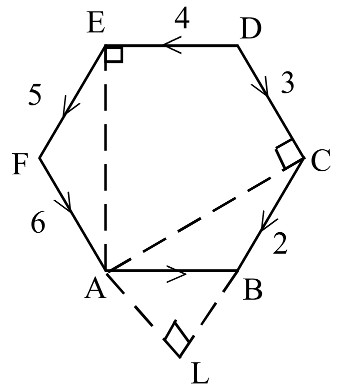
$$x = 7$$

Distance of pegs from A is 7cm & 17cm

Example 11

The side of a regular hexagon ABCDEF is 2 *m.* Forces of 1, 2, 3, 4, 5, 6N act along the sides AB, CB, DC, DE, EF and FA respectively in the order of letters. Find the sum of their moments about

(i) Vertex A (ii) centr O the hexagon



AL = 2sin60

$= \sqrt{3}m$

Sum of moments about A ○

$= 2 \times \sqrt{3} + 3 \times 2\sqrt{3} - 4 \times 2\sqrt{3} - 5 \times \sqrt{3}$

$= -5\sqrt{3}$ ○

$= 5\sqrt{3}$ Nm m

OM = 2sin60 $= \sqrt{3}m$

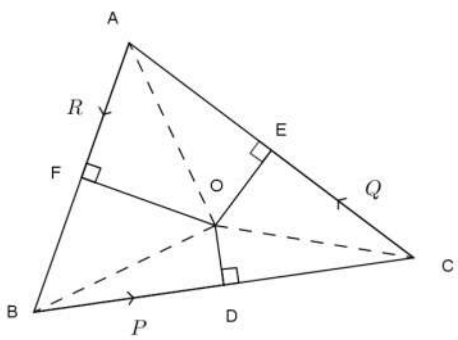
Sum of moments about O m $= 1 \times \sqrt{3} - 2 \times \sqrt{3} - 3 \times \sqrt{3} + 4 \times \sqrt{3} + 5 \times \sqrt{3} + 6 \times \sqrt{3}$

$$= 11\sqrt{3} \text{ Nm}$$

Example 12

Three forces P, Q, R act in the same sense along the sides BC, CA, AB of a triangle ABC. If the resultant passes through the circumcentre of the triangle show that

$P\cos A + Q\cos B + R\cos C = 0$



$B\hat{O}D = \hat{A}$, $C\hat{O}E = \hat{B}$

Let R' be the radius. Then R' = OA = OB = OC. ***of*** circumcentre and $A\hat{O}F = \hat{C}$
Taking moment about O

$P \times OD + Q \times OE + R \times OF = 0$

P. OBcosA + QOC. cosB + R. OAcosC = 0

since OB = OC = OA

PcosA + QcosB + RcosC = 0

3.6 Exercises

1. Masses of 1, 2, 3, 4 *kg* are suspended from a uniform rod of length 1.5 m and mass 3 kg at distances of 0.3 m, 0.6 m, 0.9 m, 1.2 m from one end. Find the position of the point about which the rod will balance.

2. A uniform beam AB of 3*m* long and mass 6 kg in supported at A and at another point on therod. A load of 1 *kg* in suspended at B, load of 5 *kg* add 4 *kg* at points 1 *m* and 2 *m* from B. If the pressure on support A is 40 N, find the position of the other support.

3. A uniform bar of 0.6 *m* long and of mass 17 *kg* is suspended by two verticle strings. One is attached at a point 7.5 *cm* from one end and just can support a weight of 7*kg* without breaking it and other string is attached 10 *cm* from other end and can just support 10 *kg* without breaking it. A weight of mass 1.7 *kg* is now attched to the rod. Find the limits of the positions in which it can be attached without breaking either string.

4. ABCD is a square of side a . Forces of 2, 3, 4 N act at A along AB, AD and AC respectively. Find the point where the line of action of the resultant meet DC.

5. Thre forces P, Q, R acting at the verticles A, B, C respectively of a triangle ABC each perpendicular to the opposite side and in equilibirium. Show that P : Q : R = a : b : c

6. Three forces P, Q, R act along the sides BC, CA and AB of a triangle ABC. If their resultant passes through the centroid show that

   (i) $\frac{P}{\sin A} + \frac{Q}{\sin B} + \frac{R}{\sin C} = 0$ (ii) $\frac{P}{BC} + \frac{Q}{CA} + \frac{R}{AB} = 0$

7. The resultant of three forces act in the same sense along the sides BC, CA, AB of a triangle ABC passes through the orthocentre and circumcentre.

   Prove that $\frac{P}{(b^2 - c^2)\cos A} = \frac{Q}{(c^2 - a^2)\cos B} = \frac{R}{(a^2 - b^2)\cos C}$

8. A system consists of three forces P, λP, λ^2P acting along the lines BC, CA, AB in the sense indicatd by the order of the letters. Show that if the resultant passes through the orthocentre of the acute angled triangle ABC then

   $$\frac{1}{\cos A} + \frac{\lambda}{\cos B} = \frac{\lambda^2}{\cos(A+B)}$$

   Deduce that λ is necessarily negative.

3.7 Couples

Definition : Two equal unlike parallel forces whose line of action are not the same form a couple.

The effect of a couple is causing rotation.

Couples are measured by their moments.

The perpendicular distance between the two lines of action is called arm of the couple.

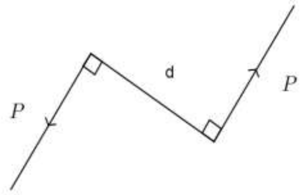
moment of a couple

The moment of a couple is the product of one of the forces and arm of the couple.

ie, moment of a couple = magnitude of a force × distance between them.

$M = P \times d$

$= Pd$ m



A couple is said to be positive or negative accroding to its tendency to cause anticlockwise or clockwis rotation.

Theorem

The algebraric sum of the moments of the two forces forming a couple about any point in their plane is constant and equal to the moment of the couple.

Proof

Let forces of the couple equal to P and O be any point in their plane. Draw OAB perpendicular to the lines of action of forces to meet at A and B.

Algebraric sum of the momens about O

= P × OB - P × OA

= P × AB m

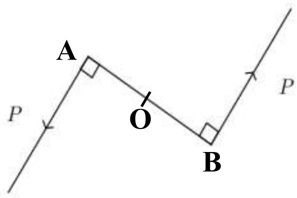
= moment of the couple

Sum of the moments about O

= P × OA + P × OB

= P(OA + AB)

= P(AB) m



Therefore moment of a couple is samewhatever the point O is taken

ie, moment of a couple is independant of the position of the point.

Theorem

Two couples acting in one plane of a rigidbody are equivalent to a single couple whose moments is the algebraic sum of the moments of the two couples.

Two cases to consider

Proof

Case (i) When the lines of action of forces are parallel let (P, P), (Q,Q) be the forces of the couples acting as shown in the diagram and OABCD is perpendicular from the point a to their lines of action. Resultant of forces P and Q acting at A and F is a force (P + Q) acting at E where AE : EF = Q : P and resultant of forces P and Q acting at B and D is (P + Q) acting at C where BC : CD = Q : P .Now equal parallel and dislike forces (P + Q) forms a single couple, which is the resultant couple of the two couple.

moment of the couple = sum of the moments of forces (P + Q) at E and (P + Q) at C about O

= sum of the moment of P at A and P at B and Q at F and Q at D

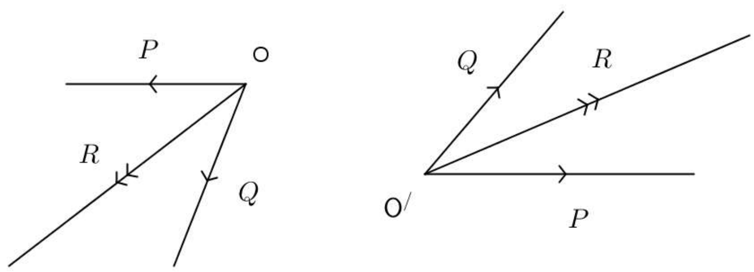
= sum of the moment of the given couple

case (ii) When the lines of action of forces are not parallel

Let P, P, Q, Q be the forces of the couples and one of the force P and one of the force Q meet at O as shown in the diagram and other forces P and Q meet at O'.

Forces P and Q at O has a resultant R at O and forces P and Q at O' has a resultant R at O'

Their resultant forces are equal parallel and dislike forces form a couple.



Moment of the couple = Moments of R at O' about O

= sum of the moments of P at O' and Q at O'

= sum of the moments of given couples

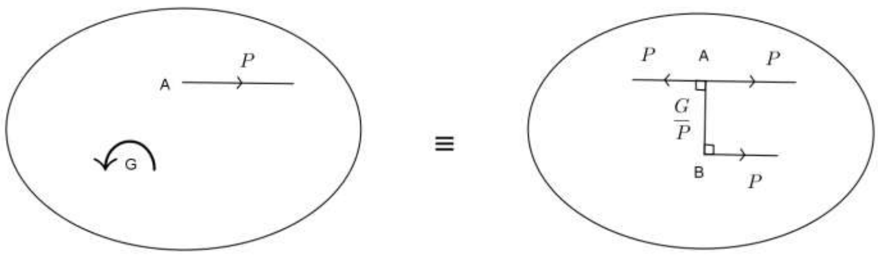
We can deduce the following

1. Two couples acting in a plance whose moments are equal and opposite balance each other.
2. Any two couples of equal moment and in the same plane are equivalent.

Theorem

Resultant of a force and a couple in the same plane, A force and a couple acting in the same place on a rigid body are equivalent to a single force equal and parallel to the given force acting on another point.

Proof



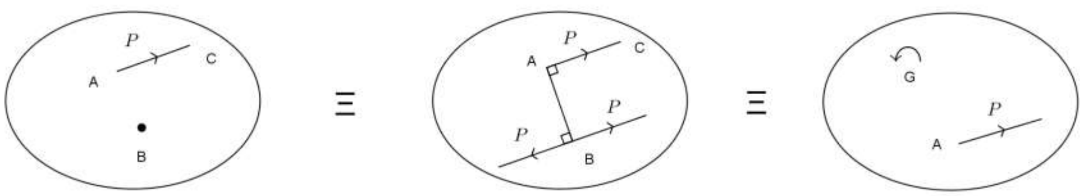
Let **P** be the single force acting at A and G be the couple acting in the same plane.

G can be replaced by two forces P and P acting at A and another point B wher AB = $\frac{G}{P}$

Now equal and opposite forces P, P at A balance each other so that the resultant is single force P at B.

Theorem

A force acting at any point of A rigidbody is equivalent to an equal and like parallel force acting at any other point together with a couple.



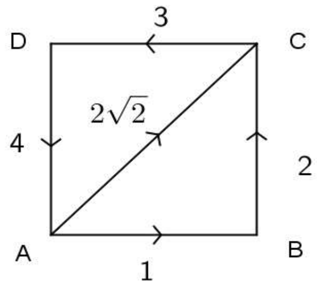
Proof

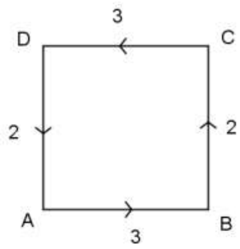
Let **P** be the given force acting at A along AC and B be any other point. Let perpendiculer distance form B to AC is d. Introduce equal and opposite parallel forces P at B. One of these forces with opposite to the P at A formes a couple G = P × d and other force at B in the single force P.

3.8 Worked examples

Example 13

ABCD is a square of side 1 m. Forces of magnitude 1, 2, 3, 4, $2\sqrt{2}$ N act along the sides AB, BC, CD, DA and diagonal AC of the square ABCD in the given order. Show that the resultant is a couple and find its moment.



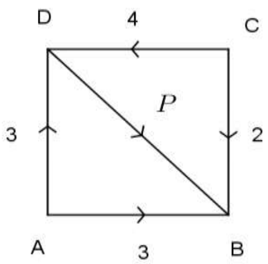


Reslove $2\sqrt{2}$ N along AD and AB. The component are $2\sqrt{2}$ cos45 = 2N.

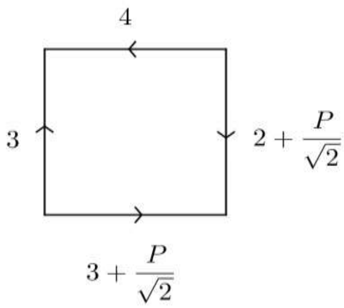
Now the system is equivalent to forces acting along the sides as shown above. The system consists of two set of parallel, equal unlike forces forms two couples, and can be combined as a single couple of moment 3 × 1 + 2 × 1 = 5 Nm m

Example 14

ABCD is a square of forces of magnitude 3, 2, 4, 3, P N act along AB, CB, CD, AD and DB respectively indicated by the order of letters. If the system reduces to a couple find the value of P.



$\equiv$



Resolve P N along AB and CB, as Pcos45 = $\frac{P}{\sqrt{2}}$ N.

To reduce to a couple, $3 + \frac{P}{\sqrt{2}} = 4$ and $2 + \frac{P}{\sqrt{2}} = 3$

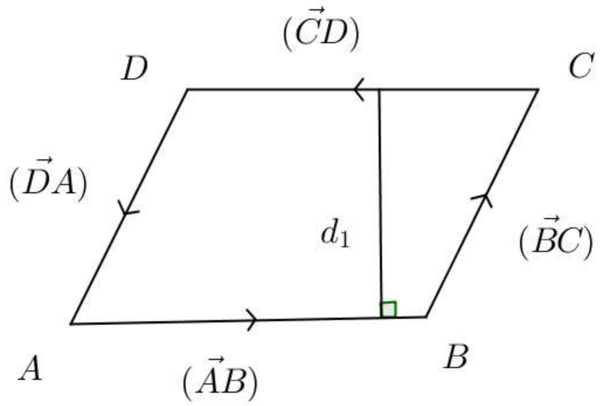
$$\frac{P}{\sqrt{2}} = 1 \text{ and } \frac{P}{\sqrt{2}} = 1$$

$$P = \sqrt{2} \text{ and } P = \sqrt{2}$$

$$\therefore\ P = \sqrt{2}$$

Example 15

ABCD is a parallelogram Forces represented by AB, BC, CD, DA act along the sides respectively in the order. Show that they are equivalent to a couple with moment numerically equal to twice the area of the parallelogram.



Forces $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal opposite and parallel forces or couple of moment. AB × d_1.
When d_1 is the distance between AB and CD.

Forces $\overrightarrow{BC}$ and $\overrightarrow{DA}$ are also equal opposite and parallel forms another couple of moment BC × d_2

also both couple acts in the same sense, so the moment of the resultant couple is

$$AB \times d_1 + BC \times d_2$$

But $AB \times d_1 = BC \times d_2$ = area of the parallelogram

Hence the moment of the couple equal to twice the area of the parallelogram.

3.9 Exercises

1. ABCD is a square of side 2 *m*. Forces a, b, c and d act along AB, BC, CD and DA taken in order and forces $p\sqrt{2}$, $q\sqrt{2}$ act along AC and BD respectively. Show that if $p+q = c - a$ and $p - q = d - b$ the forces are equivalent to a couple of moment $a + b + c + d$.

2. P and Q are two unlike parallel forces. If a couple with each forces F and whose arm is a in the plane of P and Q is combined with them. Show that the resultant is displaced through a distance $\dfrac{Fa}{(P+Q)}$

3. If three forces P, Q and R acting in the verticle of a triangle ABC along the tangents (in the same letters) to the circumcircle are equivalent to a couple. Show that

   $$P : Q : R = \sin 2A : \sin 2B : \sin 2C$$

4. ABCD is a square D and E are the midpoints of CD and BC respectively. Forces P, Q, R act along AD, DE and EA in the direction indicated by the order of letters. If the system reduces to a couple show that $P : Q : R = \sqrt{5} : \sqrt{2} : \sqrt{5}$

5. Forces 4, 3, 3 N act along the sides AB, BC, CA of a triangle ABC of an equilateral triangle ABC reprectively of side 0.6 ***m*** and another force P N acts at C so that the system is equivalent to couple. Find the magnitude, direction of P

   also find the moment of the couple.

6. If three forces acting on a rigidbody is represented magnitue direction and line of action by the three sides of a triangle taken in order show that they once equivalent to a couple of moment represented by twice the area of the triangle.

7. Four forces P, P, Q, Q act along the sides AB, BC, CD, DA of a rhombus ABCD. Find the sum of their moments about the centre O of the rhombus. Prove that their resultant is at a distance $\dfrac{BD}{2}\left(\dfrac{P+Q}{P-Q}\right)$ from O

   Discuss the case when P = Q.

4.0 Coplanar forces acting on a rigid body

4.1 Resultant of coplanar forces

In chapter two we discussed the coplanar forces acting on a point. We shall bow consider forces acting not all at one point on a rigid body.

Resultant of coplanar forces

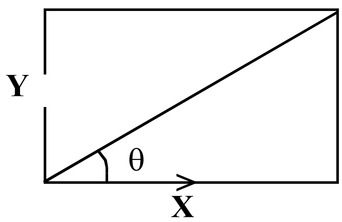
We required to find the resultant of number of forces whose magnitude and line of action are given..

The magnitude of the resultant

Resolve the forces is two directions at right angle, add these components seperately say **X** and **Y**. The magnitude of the resultant is obtained by $R^2 = X^2 + Y^2$

The direction of the resultant

If the angle made with the direction of X is θ.



Then $\quad \tan\theta = \frac{Y}{X}$

$$\theta = \tan^{-1}\left(\frac{Y}{X}\right)$$

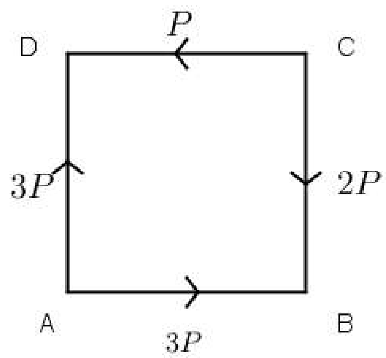
To find the position of the line of action.
By taking moments about any point O given line we can find where the line of action of the rsultant cuts the line.

Example 1

ABCD is a square of side ***2a.*** Forces 3P, 2P, P, 3P Newtons act along the sides AB, CB, CD, AD respectively indicated by the order of letters.

Find (i) The nagnitude and direction

(ii) The line of action of resultant



Resolving Parallel to AB $\rightarrow$ $\quad X = 3P - P = 2P$

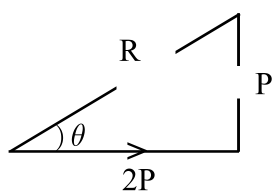
Resolving parallel AD $\uparrow$ $\quad Y = 3P - 2P = P$

$$R^2 = X^2 + Y^2$$

$$= (2P)^2 + P^2 = 5P^2$$

$$R = P\sqrt{5}\ N$$

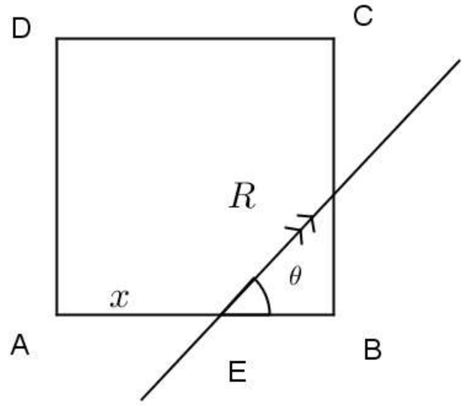
$$\tan\theta = \frac{Y}{X} = \frac{P}{2P} = \frac{1}{2}$$



$$\theta = \tan^{-1}\left(\frac{1}{2}\right)$$

Resultant is of magnitude $P\sqrt{5}$ N, makes $\tan^{-1}\left(\frac{1}{2}\right)$ with AB. Let the resultant cuts AB at E where AE = x

Taking moment about E,



$$\begin{aligned} 3Px + 2P(2a - x) - P \times 2a &= 0 \\ 3x - 2x &= 2a - 4a \\ x &= -2a \end{aligned}$$

OR

taking moments about A

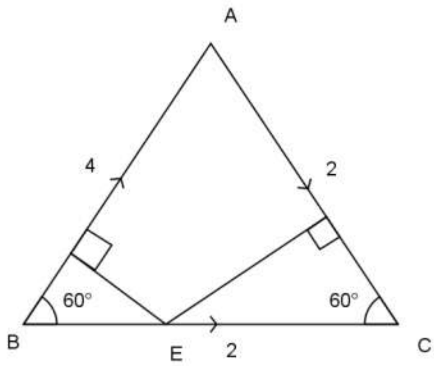
$$\begin{aligned} R \times x\sin\theta &= P \times 2a - 2P \times 2a \\ P\sqrt{5} \times x \times \frac{1}{\sqrt{5}} &= -2Pa \\ x &= -2a \end{aligned}$$

Resultant cuts BA produced at a distance 2a from A.

Example 2

ABC is an equilateral triangle of side 2a. Forces 4, 2, 2, Newtons act along the sides BA, AC, BC in the directions indicated by the order of letters.

Find the magnitude of the resultant and show that the line of action cuts BC at a distance $\frac{2a}{3}$ from B.

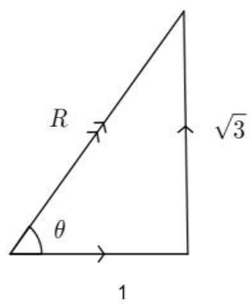


Resolving parallel to BC

$$\begin{aligned} \rightarrow \quad X &= 2 + 2\cos 60 - 4\cos 60 \\ &= 2 + 1 - 2 = 1 \end{aligned}$$

Resolving perpendicular to BC

$$\begin{aligned} \uparrow \quad Y &= 4\sin 60 - 2\sin 60 \\ &= 2 \times \frac{\sqrt{3}}{2} = \sqrt{3} \end{aligned}$$



$$\begin{aligned} R^2 &= X^2 + Y^2 \\ &= 1^2 + (\sqrt{3})^2 \\ &= 1 + 3 = 4 \\ R &= 2N \end{aligned}$$

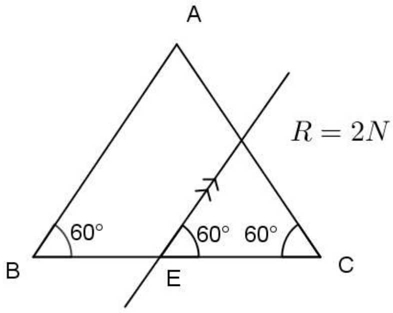
$$\tan\theta = \frac{Y}{X} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\theta = \tan^{-1}(\sqrt{3}) = 60°$$

Magnitude of resultant 2N, making 60° with BC

Suppose that the resultant cuts BC at E

Taking moments about E



$4 \times x\sin 60 - 2 \times (2a - x)\sin 60° = 0$

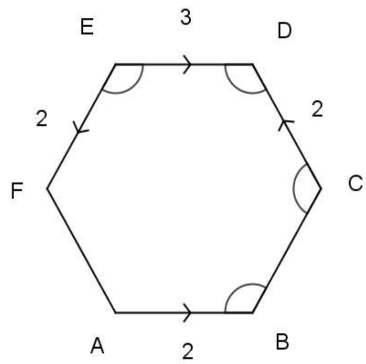
$4x - 4a + 2x = 0$

$6x = 4a$

$x = \frac{2}{3}a$

Example 3

ABCDEF is a regular hexagon of side a metres. Forces of 2, 2, 3, 2 Newton act along the sides AB, CD, ED, EF respectively indicated by the order of letters. Find the magnitude of the resultant and show that it acts through A along AB.



AB and AE are perpendicular to each other.

Resolve the forces parallel to AB

$$\rightarrow X = 2 + 3 - 2\cos 60 - 2\cos 60$$
$$= 3$$

Resolving parallel to AE

$$Y = 2\sin 60 - 2\sin 60$$
$$= 0$$

$R = 3N$ parallel to AB

Taking moment about A

m $2 \times 2a\sin 60 - 3 \times 4a\cos 30 + 2 \times 4a\cos 30$
$$= 0$$

It acts through A along AB

Reducing a system of coplanar forces

Any system of coplanar forces acting on a rigid body can, in general, be reduced to a single force acting at an arbitrary point in the plane of the forces together with a couple.

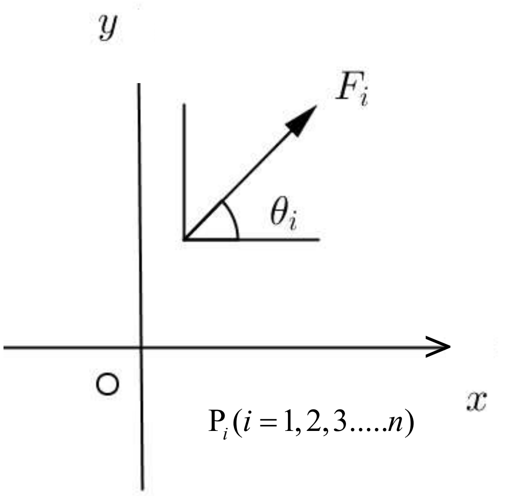
Let the forces F_i $(i = 1, 2, ... n)$ act at points P_i $(i = 1, ... n)$ in a plane and O be any point in the plane of the forces. Take O as origin of coordinate axes refered to rectangular axes Ox, Oy coordinate of $P_i \equiv (x_i, y_i)$ $(i = 1, 2, ... n)$

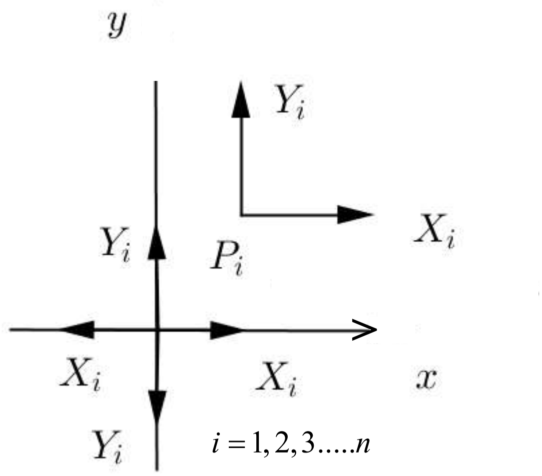
Let forces F_i $(i=1,2,3,....n)$ makes an angle θ_i with Ox axis

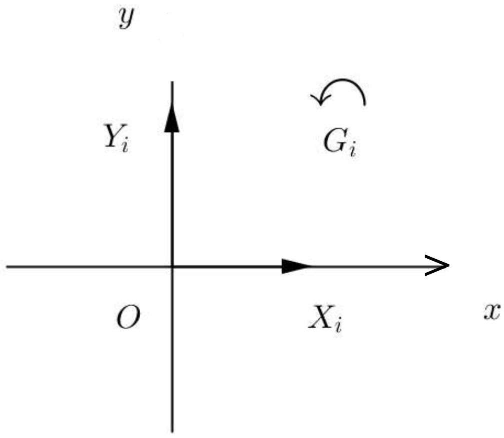
Resolve force F_i into components X_i, Y_i.

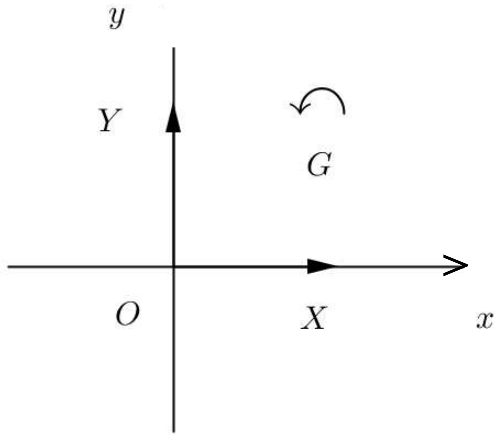
Where $X_i = F_i\cos\theta_i$, $Y_i = F_i\sin\theta_i$ $(i = 1, 2,......n)$

Inroduce equal and opposite forces X_i, Y_i at O. This has no effect in the given system of forces,

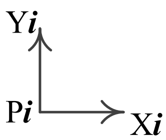




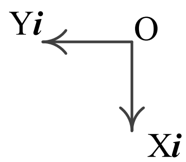




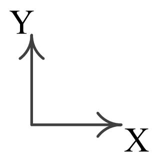
Now at



and at O



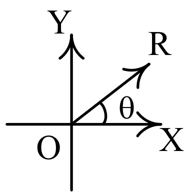
forms a couple of moment G_i and at O the force is



G_i m $= Y_i x_i - X_i y_i$

Let $X = \sum_{i=1}^{n} X_i$

$Y = \sum_{i=1}^{n} Y_i$



Then $\quad R^2 = X^2 + Y^2$

$$\tan\theta = \frac{Y}{X}$$

$$\theta = \tan^{-1}\left(\frac{Y}{X}\right)$$

$$\text{and} \quad G = \sum_{i=1}^{n} Y_i x_i - X_i y_i$$

Note : G is the sum of the moments of all forces in the given system about O, and will depend on the position of O.

Conditions of equilibirium of a system of coplanar forces

Any system of forces can be reduced to a single force R at any arbitrary point O (origon) together with a couple G in the plane in general.

If,

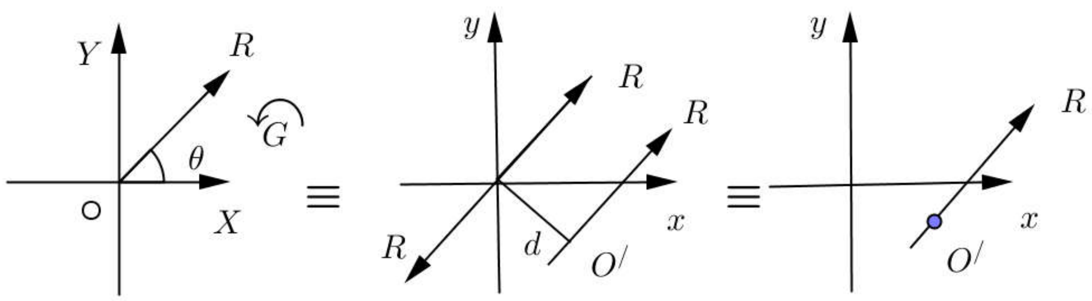
i. $R = 0$ and $G = 0$ the system is in equilibirium

ii. $R \neq 0$, $G = 0$ the system reduces to a single force acting at O

iii. $R = 0$ and $G \neq 0$ the system reduces to a couple of moment G

iv. $R \neq 0$ and $G \neq 0$ the system is not in equilibirium and can be reduced to a single force R acting at another point O'

Where $OO' = \frac{G}{R}$ as shown below.



proof :

Replace couple G by equal unlike parallel forces R and R acting at O and O' where distance between the lines of action $d = \frac{G}{R}$

equal and opposite forces balance each. results single force R at O'

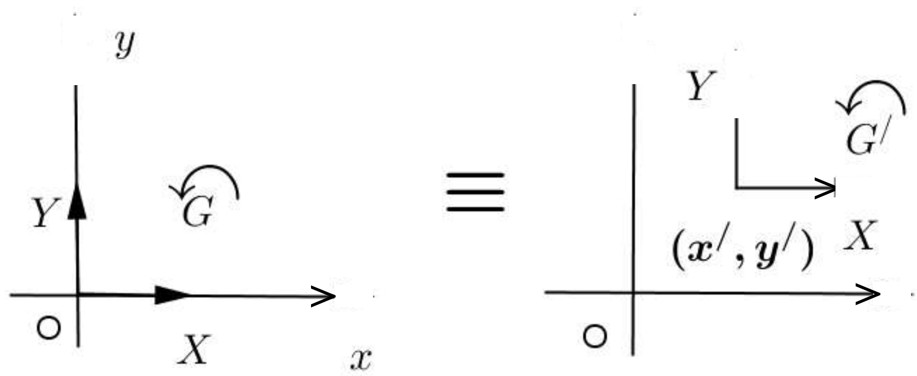
Equation of line of action of the resultant of a system of forces

If a system of forces is not in equilibirium and it can be reduced to a single force R at any point (x', y') together with a couple G'

Then $R^2 = X^2 + Y^2$ and $G' = G + Xy' - Y x'$

Moment about any point in the line of action of the resultant is zero. Let (x, y) be any point on the line of action Then $G + Xy - Yx = 0$;

This is the equation of line of acting of the resultant.



Moment about O =Moment about O

$$G = Yx' - Xy' + G'$$

$$\text{Hence } G' = G + Xy' - Yx'$$

If the resultant passes through (x', y') then $G' = 0$

$$O = G + Xy' - Yx'$$

and the equation of line of action is

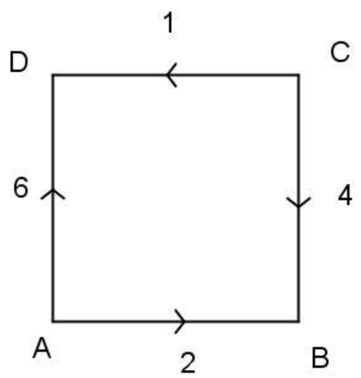
$$O = G + Xy - Yx$$

4.2 Worked examples

Example 4

Forces 2, 4, 1, 6 N act along the sides AB, CB, CD, AD of a square respectively. Find the magnitude and direction of the resultant.

Prove that the equation of the line of action of the resultant referred to AB and AD as coordinate axis is $2x - y + 3a = 0$



Resolving parallel to AB;

$$\rightarrow X = 2 - 1 = 1$$

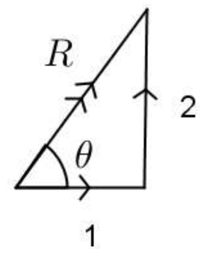
Resolving parallel to AD,

$$\uparrow Y = 6 - 4 = 2$$

$$R^2 = X^2 + Y^2 = 2^2 + 1^2 = 5$$

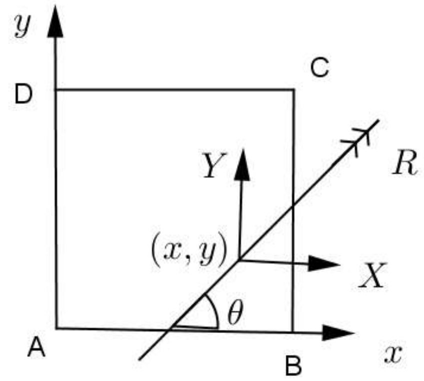
$$R = \sqrt{5}\text{ N}$$

$$\tan\theta = \frac{X}{Y}$$
$$= 2$$
$$\theta = \tan^{-1}(2)$$



Resultant of magnitude $\sqrt{5}$ N, makes $\tan^{-1}(2)$ with AB.
Taking moments about A,

m $G = 1 \times a - 4 \times a$
$= -3a$ Nm



Equation of line of action is

$G + Xy - Yx = 0$
$-3a + y - 2x = 0$
$2x - y + 3a = 0$

OR

Moment of the resultant about A
= Algebraic sum of the moments of the forces about A

$$G = Yx - Xy$$
$$-3a = 2x - 1y$$
$$2x - y + 3a = 0$$

Example 5

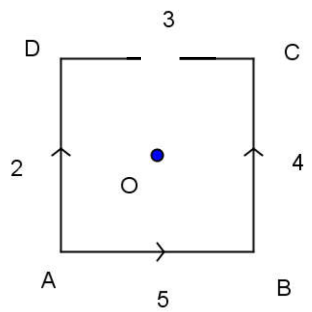
ABCD is a square of side a. Forces of 5, 4, 3, 2 N act along the sides AB, BC, CD, AD is the direction indicated bu the order of letters. Reduce the system to

i. a single force at A with a couple.

ii. a single force at the centre O with a couple.

iii. Refered to AB and AD as axis find the equation of line of action.

Resolving parallel to AB

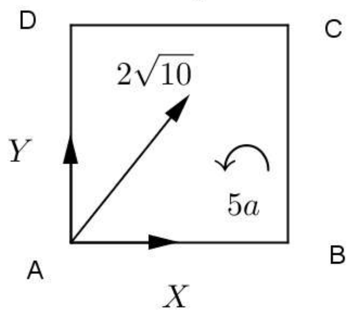


$\rightarrow X = 5 - 3$
$= 2$

Resolving parallel to AB

$\uparrow Y = 4 + 2$
$= 6$

$$R^2 = X^2 + Y^2$$
$$= 40$$
$$R = 2\sqrt{10}$$



moment about A

$$G = 4 \times a + 3 \times a$$
$$= 7a \text{ Nm}$$

single force of $2\sqrt{10}$ N with a couple of $7a$ Nm at A

Moment about O,

$$\begin{aligned} G &= 5 \times \frac{a}{2} + 4 \times \frac{a}{2} + 3 \times \frac{a}{2} - 2 \times \frac{a}{2} \\ &= 5a \text{ N}m \end{aligned}$$

At the centre, $2\sqrt{10}$ N force with a couple $5a$ Nm.
Equation of line of action

$$\begin{aligned} G + X.y - Y.x &= 0 \\ 5a + 2y - 4x &= 0 \\ 4x - 2y - 5a &= 0 \end{aligned}$$

Example 6

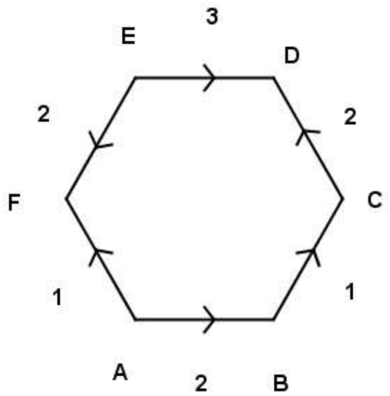
ABCDEF is a regular hexagon of side $2a$. Forces of 2, 1, 2, 3, 2, 1 N act along the sides AB, BC, CD, ED, EF, AF in the direction indicated by the order of letters respectively

i. show that the system can be reduced to a force of magnitude $2\sqrt{3}$N along AD with a couple. Find the moment of the couple.

ii. Show that the system can be reduced to a single force and find the equation of its line of action.

iii. If the line of action cuts FA produced at K find the length of AK.

$\rightarrow$ $X = 2 + 3 + 1\cos 60 - 2\cos 60 - 2\cos 60 - 1\cos 60$
$= 5 - 2 = 3$ N



$\uparrow$ $Y = 1\sin 60 + 1 \sin 60 + 2\sin 60 - 2\sin 60$

$= \sqrt{3}$N

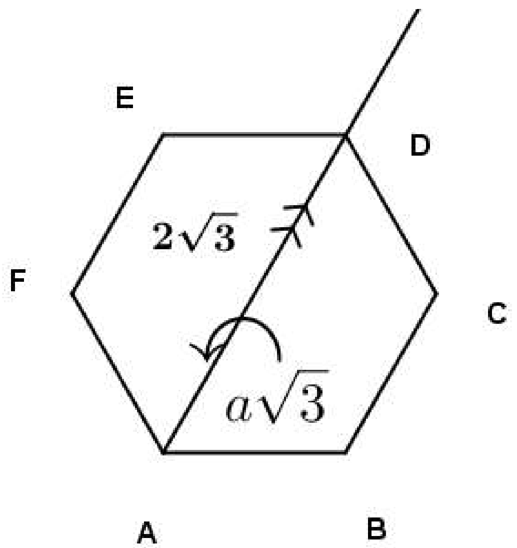
$R^2 = X^2 + Y^2$
$= 12$

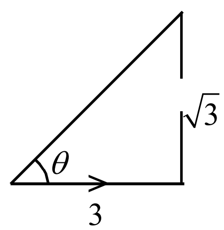
$R = 2\sqrt{3}$N

$\tan\theta = \dfrac{X}{Y} = \dfrac{1}{\sqrt{3}}$

$\theta = 30°$

Resultant is parallel to AD





Sum of moment of the forces about A

$G = 1 \times 2a\sin60 + 2 \times 4a\cos 30 + 2 \times 2a\sin60 - 3 \times 4a\cos30$

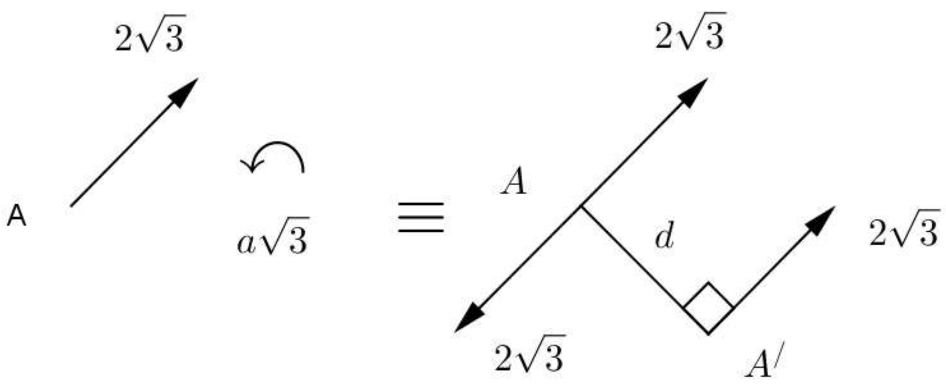
$= a\sqrt{3} + 4\sqrt{3}\,a + 2\sqrt{3}\,a - 6\sqrt{3}\,a$

$= a\sqrt{3}$ Nm

system can be reduced to G

force of $2\sqrt{3}$ N along AD with a couple of moment a $\sqrt{3}$ N

moment of the couple a $\sqrt{3}$ Nm



$2\sqrt{3}$ Nm can be replaced by equal opposite parallel forces of $2\sqrt{3}$ N shown above where

$AA' = \frac{a\sqrt{3}}{2\sqrt{3}} = \frac{a}{2}m$

The forces at A balance each other,

Therefore it reduces to a single force $2\sqrt{3}$ N at A'

equation of line of action

$$G + Xy - Yx = 0$$

$$a\sqrt{3} + 3y - \sqrt{3}x = 0$$

$$x - \sqrt{3}y - a = 0$$

The line of action cuts AB at H at FA produced at K

Let $E = (x_1, 0)$ $\quad x - \sqrt{3}\text{y} - a = 0$

$y = 0 \quad x_1 = a$

$E = (a, 0)$

$$\sin60 = \frac{AK}{AE}$$

$$AK = AE\sin60$$

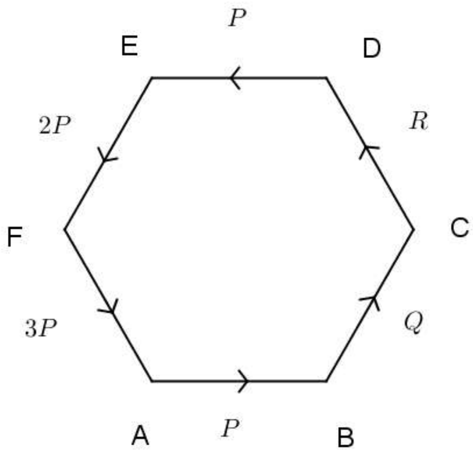
$$= \frac{a\sqrt{3}}{2}m$$

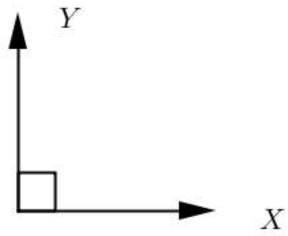
Example 7

Forces P, Q, R, P, 2P, 3P N act along the sides AB, BC, CD, DE, EF, FA respectively of a regular hexagon ABCDEF of side $2a$ metres in the sense indicated by the order of letters.

i. If the system is equivalent to a couple show that Q = 2P and R = 3P and calculate the moment of the couple.

ii. If the system is equivalent to a single force along AD find Q and R in terms of P.





System is equivalent to a couple

X = 0 and Y = 0

$$\rightarrow\ X = Q\cos60 - R\cos60 - 2P\cos60 + 3P\cos60 = \frac{Q-R+P}{2}$$

$$X = 0;\quad Q - R + P = 0$$

$$R - Q = P \qquad \text{.............................(1)}$$

$$\uparrow\ Y = Q\sin60 + R\sin60 - 2P\sin60 - 3P\sin60 = (Q+R-5P)\frac{\sqrt{3}}{2}$$

$$Y = 0;\quad Q + R = 5P \qquad \text{..............................(2)}$$

(1), (2) $\quad R = 3P$ and $Q = 2P$

Moment of the couple = sum of the moments about O

$$= (P + Q + R + P + 2P + 3P) \times a\sqrt{3}$$

$$= 12\sqrt{3}\,aP \text{ N}m \text{ m}$$

(ii)
$$X = \frac{Q-R+P}{2}$$

$$Y = (Q + R - 5P)\frac{\sqrt{3}}{2}$$

Resultant parallel to AD

$$\theta = 60°$$

$$\tan\theta = \sqrt{3} = \frac{\sqrt{3}(Q+R-5P)}{Q-R+P}$$

$$Q - R + P = Q + R - 5P$$

$$R = 3P$$

Single force along AD

Sum of moments about O is zero,

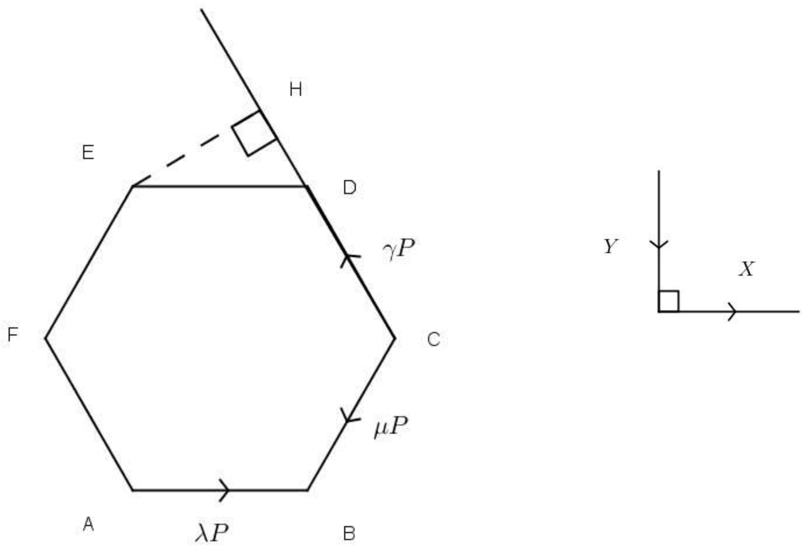
$$\begin{aligned}(7P + Q + R)a\sqrt{3} &= 0\\ 10P + Q &= 0\\ Q &= -10P\end{aligned}$$

Example 8

ABCDEF is a regular hexagon of side a forces of magnitude λP, μP, γP act along the sides AB, CB, CD in the sense indicated by the order of letters respectively. The algebraic sum of the moments about verticle D, E, F are $2\sqrt{3}$ Pa, $\frac{3\sqrt{3}}{2}$ Pa, $\frac{\sqrt{3}}{2}$ Pa respectively in anticlockwise direction.

i. Find the values of λ, μ, γ.

ii. show that the resultant is a single force, parallel to EC through A of magnitude of $\sqrt{3}$ PN.



Moment about D

$$\text{m}\ \lambda P \times 2a\cos 30 - \mu P \times a\sin 60 = 2\sqrt{3}\,aP$$

$$2\lambda\frac{\sqrt{3}}{2} - \mu\frac{\sqrt{3}}{2} = 2\sqrt{3}$$

$$2\lambda - \mu = 4 \qquad \text{.........................(1)}$$

Sum of moments about E

$$\text{m}\ \lambda P \times 2a\cos 30 - \mu P \times 2a\cos 30 + \gamma P \times a\sin 60$$

$$aP\frac{\sqrt{3}}{2}[2\lambda - 2\mu + \gamma] = 3a\frac{P\sqrt{3}}{2}$$

$$2\lambda - 2\mu + \gamma = 3 \qquad \text{...........................(2)}$$

AE = BF = CE $= 2a\cos 30 = a\sqrt{3}$

$EH = a\sin 60 = a\frac{\sqrt{3}}{2}$

Sum of moments about F

$$\lambda P a\frac{\sqrt{3}}{2} - \mu P 2a\frac{\sqrt{3}}{2} + \gamma P 2a\frac{\sqrt{3}}{2}$$

$$ap\frac{\sqrt{3}}{2}[\lambda - 2\mu + 2\gamma] = a\frac{P\sqrt{3}}{2}$$

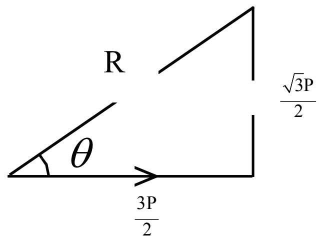
$$\lambda - 2\mu + 2\gamma = 1 \quad \text{........................} \quad (3)$$

(1), (2), (3) $\lambda = 3, \mu = 2, \gamma = 1$

$$\rightarrow X = \lambda P - \mu P\cos 60 - \gamma P\cos 60$$

$$= 3P - 2\frac{P}{2} - \frac{P}{2} = \frac{3P}{2}$$

$$\uparrow Y = \mu P\sin 60 - \gamma P\sin 60 = P\frac{\sqrt{3}}{2}$$



$$R^2 = \left(3\frac{P}{2}\right)^2 + \left(P\frac{\sqrt{3}}{2}\right)^2 = 3P^2$$

$$R = P\sqrt{3}\ N$$

$$\tan\theta = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}} \quad \theta = 30°$$

Moment about A m $P \times 2a\cos 30 - 2P \times a\cos 60$

$$= 0$$

Resultant is a single force through A of magnitute $P\sqrt{3}$ N as in parallel to EC.

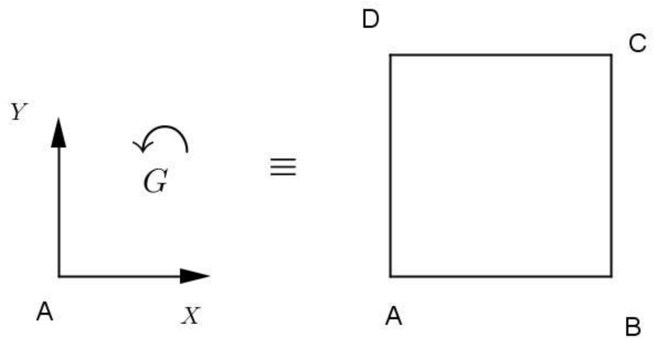
Example 9

ABCD is a rectangle with AB = $2a$, AD = $2a$. The moment of a system of forces in the plane of the rectangle are M_1, M_2, M_3 about points A, B and C respectively

i. Find the moment of the system about D.

ii. Determine the magnitude and direction of the resultant of the system.

iii. Find the equation of line of action of the resultant and if the line of action is perpendicualr to BC show that $M_1 = 5M_2 + 4M_3$



Let the system of forces reduced at A as shown

Moment of a system about any point (x, y) in its plane

$$G' = G + Xy - Yx$$

Let A = (0, 0) , B = (2a, 0) , C = (2a, a) , D = (0, a)

Then about A

$$M_1 = G + X.0 - Y.0$$

$$G = M_1$$

about B

$$M_2 = M_1 + X(0) - Y(2a)$$

$$Y = \frac{M_1 - M_2}{2a}$$

about C

$$-M_3 = M_1 + X(x) - Y(2a)$$

$$-M_3 = M_1 + Xa - (M_1 - M_2)$$

$$X = -\frac{(M_2 + M_3)}{a}$$

moment about D = (0, a)

$$G' = M_1 - \frac{(M_2 + M_3)}{a} \times a - \left(\frac{M_1 - M_2}{2a}\right) \times 0$$

$$= (M_1 - M_2 - M_3)$$

$$R^2 = X^2 + Y^2$$

$$= \left(\frac{M_2 + M_3}{a}\right)^2 + \left(\frac{M_1 + M_2}{2a}\right)^2$$

$$= \frac{4(M_2 + M_3)^2 + (M_1 - M_2)^2}{4a^2}$$

$$R = \frac{1}{2a}\left[(M_2 + M_3)^2 + (M_1 - M_2)^2\right]^{\frac{1}{2}}$$

$$\tan\theta = \frac{Y}{X}$$

$$= \left(\frac{M_1 - M_2}{2a}\right) \times \left(\frac{-a}{M_2 + M_3}\right) = \frac{1}{2}\left(\frac{M_2 - M_1}{M_2 + M_3}\right)$$

$$\theta = \tan^{-1}\left[\frac{M_2 - M_1}{2(M_2 + M_3)}\right]$$

Equation of line of action

$$G + Xy - Yx = 0$$

$$M_1 - \frac{(M_2 + M_3)}{a}y - \left(\frac{M_1 - M_2}{2a}\right)x = 0$$

$$(M_1 - M_2)x + 2(M_2 + M_3)y - 2aM_1 = 0$$

$$\text{Gradient of the line} = \frac{(M_2 - M_1)}{2(M_2 + M_3)}$$

$$\text{Gradient of AC} = \frac{1}{2}$$

$$\frac{(M_2 - M_1)}{2(M_2 + M_3)} \times \frac{1}{2} = -1$$

$$M_2 - M_1 = -4M_2 - 4M_3$$

$$5M_2 + 4M_3 = M_1$$

Example 10

A system of forcs F_i ($\boldsymbol{i}$ = 1, 2, ... n) act at points P_i whose coordinates (x_i, y_i) related to rectangular axis Ox, Oy. Each force of the system makes an angle θ with Ox.

i. Reduce the system as a single force at O together with a couple.

ii. Write down the equation of the line of action of the resultant

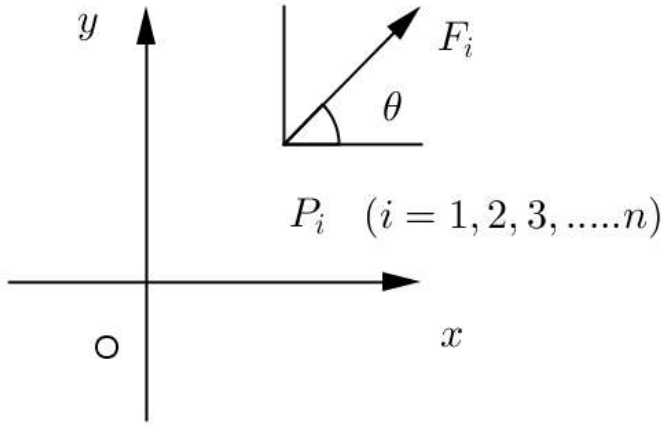
iii. Reduce as θ varies, the corresponding resultant of the system passes through a fixed point in the plane and find its coordinate.

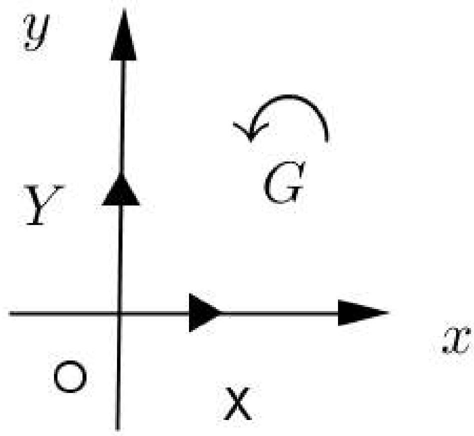
$$X_i = F_i \cos\theta \qquad \boldsymbol{i} = (1, 2, \ldots n)$$

$$X = \sum_{i=1}^{n} X_i = \cos\theta \sum_{i=1}^{n} F_i$$

$$Y_i = F_i \sin\theta \qquad \boldsymbol{i} = (1, 2, \ldots n)$$

$$Y = \sum_{i=1}^{n} Y_i = \sin\theta \sum_{i=1}^{n} F_i$$





Moment of a Force Fi about O is

$$G_i = Y_i x_i - X_i y_i$$

Sum of the moments

$$G = \sum_{i=1}^{n} G_i$$

$$= \sum_{i=1}^{n} \left(Y_i x_i - X_i y_i \right)$$

$$= \sum_{i=1}^{n} F_i x_i \; - \sum_{i=1}^{n} F_i y_i$$

$$R^2 = X^2 + Y^2$$

$$= \cos^2\theta \left(\sum_{i=1}^{n} F_i \right)^2 + \sin^2\theta \left(\sum_{i=1}^{n} F_i \right)^2$$

$$= \left(\sum_{i=1}^{n} F_i \right)^2$$

$$R = \sum_{i=1}^{n} F_i$$

equation of line of action

$$G + Xy - Yx = 0$$

$$\sin\theta \sum_{i=1}^{n} F_i x_i \; - \cos\theta \sum_{i=1}^{n} F_i y_i \; + y\cos\theta \sum_{i=1}^{n} F_i \; - x\sin\theta \sum_{i=1}^{n} F_i = 0$$

$$\sin\theta\left(\sum_{i=1}^{n} F_i x_i - x\sum_{i=1}^{n} F_i \right) - \cos\theta \left(\sum_{i=1}^{n} F_i y_i - y\sum_{i=1}^{n} F_i \right) = 0$$

$$x\sin\theta \sum_{i=1}^{n} F_i - y\cos\theta \sum_{i=1}^{n} F_i + \cos\theta \sum_{i=1}^{n} F_i y_i - \sin\theta \sum_{i=1}^{n} F_i x_i = 0$$

This is a variable line depends on θ.

as θ varies,

The line passes through a point

where $x = \frac{\sum_{i=1}^{n} F_i x_i}{\sum_{i=1}^{n} F_i}$ $\quad y = \frac{\sum_{i=1}^{n} F_i y_i}{\sum_{i=1}^{n} F_i}$

independent of θ.

Edit of the fixed point $\left(\frac{\sum_{i=1}^{n} \mathrm{F}_i x_i}{\sum_{i=1}^{n} \mathrm{F}_i}, \frac{\sum_{i=1}^{n} \mathrm{F}_i y_i}{\sum_{i=1}^{n} \mathrm{F}_i} \right)$ OR

$$\sin\theta\left(\sum_{i=1}^{n} \mathrm{F}_i x_i - x\sum_{i=1}^{n} \mathrm{F}_i\right) - \cos\theta\left(\sum_{i=1}^{n} \mathrm{F}_i y_i - y\sum_{i=1}^{n} \mathrm{F}_i\right) = 0$$

is a straight line passing through the point of intersection of two lines

$\sum_{i=1}^{n} \mathrm{F}_i x_i - x\sum_{i=1}^{n} \mathrm{F}_i = 0$ and $\sum_{i=1}^{n} \mathrm{F}_i y_i - y\sum_{i=1}^{n} \mathrm{F}_i = 0$

for all values of θ

Hence point of intersection is $\left(\frac{\sum_{i=1}^{n} \mathrm{F}_i x_i}{\sum_{i=1}^{n} \mathrm{F}_i}, \frac{\sum_{i=1}^{n} \mathrm{F}_i y_i}{\sum_{i=1}^{n} \mathrm{F}_i} \right)$

Example 11

Forces P, 7P, 8P, 7P, 3P newtons act along the sides AB, CB, CD, ED and FE respectively of a regular hexagon ABCDEF of side a meters in the direction indicated by the order of letters.

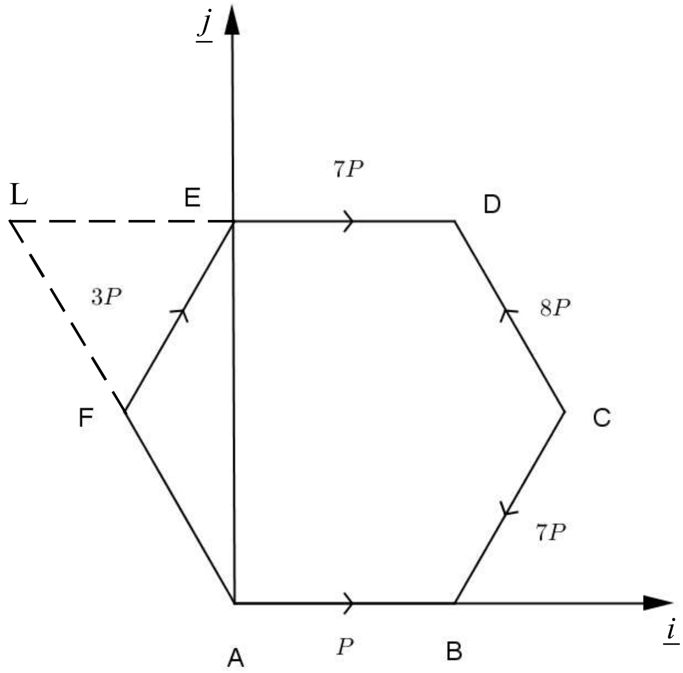
Taking $\underline{i}$ and $\underline{j}$ be the unit vectors along the directions $\overrightarrow{AB}$ and $\overrightarrow{AD}$ respectively express each force in terms of $\underline{i}, \underline{j}$ and P. Show that the given system is equivalent to a single force

$\underline{R} = 2P(\underline{i} + \sqrt{3}\,\underline{j})$ parallel to $\overrightarrow{BC}$.

What is the magnitude of $\underline{R}$

Show further that the line of action of the resultant passes through the common point of DE and AF (both produced)

If the system is equivalent to a force R acting hrough A together with a couple find the moment of this couple.



Forces are,

$P\underline{\boldsymbol{i}}$ along $\overrightarrow{AB}$

$7P\left(-\frac{1}{2}\underline{\boldsymbol{i}}-\frac{\sqrt{3}}{2}\underline{\boldsymbol{j}}\right)$ along $\overrightarrow{CE}$

$8P\left(-\frac{1}{2}\underline{\boldsymbol{i}}+\frac{\sqrt{3}}{2}\underline{\boldsymbol{j}}\right)$ along $\overrightarrow{CD}$

$7P\underline{\boldsymbol{i}}$ along $\overrightarrow{ED}$

$3P\left(\frac{1}{2}\underline{\boldsymbol{i}}-\frac{\sqrt{3}}{2}\underline{\boldsymbol{j}}\right)$ along $\overrightarrow{FE}$

$$\text{Resultant } \underline{R} = \left(1-\frac{7}{2}-\frac{8}{2}+7+\frac{3}{2}\right)P\underline{i} + \left(-\frac{7\sqrt{3}}{2}+\frac{8\sqrt{3}}{2}+\frac{3\sqrt{3}}{2}\right)P\underline{j}$$

$$= 2P\underline{i} + 2\sqrt{3}\,P\underline{j}$$

$$= 2P\left(\underline{\boldsymbol{i}}+\sqrt{3}\,\underline{\boldsymbol{j}}\right) \text{ is single resultant force}$$

$$\overrightarrow{BC} = \frac{a}{2}\underline{\boldsymbol{i}}+\frac{a\sqrt{3}}{2}\underline{\boldsymbol{j}}$$

$$= \frac{a}{2}\left(\underline{\boldsymbol{i}}+\sqrt{3}\,\underline{\boldsymbol{j}}\right)$$

$\therefore$ $\underline{\mathbf{R}}$ is parallel to $\overrightarrow{BC}$

$$|\underline{R}| = \sqrt{(2P)^2 + (2\sqrt{3}P)^2}$$
$$= 4P$$

Taking moment about L,

$P \times 2a\cos30 - 7P \times 3a\cos30 + 8P \times 2a\sin60 + 3P \times a\sin60$

$$21Pa\sin60 - 21Pa\cos30 = 0$$

resultant pass through the intersection point of DE, AF prodeced.

4.3 Exercises

1. Forces 1, 3, 5, 7, $9\sqrt{2}$ act along the sides AB, BC, CD, DA and the diagonal BD of a square ABCD of side a in the sense indicated by the order of letters. Taking AB and AD as axes of *x* and *y* respectively find
   i. Magnitude and direction of the resultant
   ii. Equation of the line of action of the resultant
   iii. The point in which the resultant cuts AB.

2. ABCDEF is a regular hexagon of side *a*. Forces 1, 3, 2, 4 N act along AB, BE, ED and DA respectively indicated by the order of letters. Taking AB and AD as *x* and *y* axes respectively, find
   i. magnitude and direction of resultant
   ii. equation of line of action.

3. Forces of magnitude F, 2F, 3F, 4F, 5F, 6F act along the sides AB, BC, CD, DE, DF, FA of a regular hexagon of side *a* taken inorder. Show that
   i. They are equivalent to a single force 6F acting parallel to one of the given forces.
   ii. The distance of the line of action of that force and of the resultant from the centre of the hexagon is in the ratio 2 : 7

4. Forces 4, 3, 3 N act along the sides AB, BC, CA of a equilateral triangle ABC of side 0.6 m indicated by the order of leters. Find
   i. The magnitude and direction of the resultant
   ii. The perpendicular distance of the line of action from C
   iii. If an additional force F is introduced at C in the plane of ABD, the system is now equivalent to a couple. Find the moment of the couple and magnitude and direction of the force introduced.

5. The coordinates of the points O, A, B, C are (0, 0), (3, 0), (3, 4) and (0, 4) respectively. Forces of magnitude 7, 6, 2, 9, 5 N act along CA, AB, BC, CO, OB indicated by the order of letters and a couple of moment 16 units acts in the plane in the sense OCBA. Reduce the given system to a force and a couple at O.
Show that the system is equivalent to a single force acting along the line $3x - 4y - 5 = 0$.

6. ABCDEF is a reguler hexagon of centre O and length of a side a metres. Five forces P, 2P, 3P, 4P, 5P Newton act along the sides AB, BC, CD, DE, EF respectively in the direction indicated by the order of letters. Three new forces Q, R, S newtons acting along AF, FO, OA respectively are added to the system. Find the values of Q, R, S in terms of P.

   i. if the whole system is in equilibirum

   ii. equivalent to a couple of moment $Pa\sqrt{3}$ Nm in the tense ABC.

7. The points A, B, C, D, E, F are vertices of a regualr hexagon of side $2a$ m in anticlockwise sense. Forces of magnitudes P, 2P, P, mP, nP and 2P newtons act along the sides AB, CB, DC, DE, FE and FA respectively in the sense indicated by the order of letters.

   i. If the system reduces to a single force acting along DA find the values of m and n.

   ii. A clockwise couple of magnitude $2\sqrt{3}\,Pa$ Nm in the place of the hexagon is added to the single show that the new system reduces to a single force and find the point of its line of action with AB produced if necessary.

8. Let ABCD be a square of side a metres. Forces of magnitudes 4, $6\sqrt{2}$, 8, 10, X and Y newtons act along AD, CD, AC, BD, AB and CB respectively, in directions indicated by the order of letters. The system reduces to a single resultant acting along $\overrightarrow{OE}$, where O and E are the midpoints of AC and CD respectively. Find the values of X and Y, and show that the magnitude of the resultant is 4K newtons, where $K = 2 - \sqrt{2}$.

   Let F be the point such that OAFD is a square. Find the two forces, one along $\overrightarrow{AD}$ and the other through the point F, which are equivalent to the above system.
   A couple of moment 6 ka newton metres acting in the sense ABCD, in the plane of the forces, is added to the original system. Find the line of action of the new system.

9. ABC is an equilateral triangle; O is the centre and R is the radius of the circumcircle of the triangle ABC. A system consists of six forces of magnitudes L, L, M, M and N, N acting along BC, OA, CA, OB, AF and OC respectively in the sence indicated by the order of the letters and a non-zero couple of moment $\lambda R(L + M + N)$ acting in the plane of the triangle ABC, in the sense ACB.

   Show that if the system reduces to

   a. a single force, then $L^2 + M^2 + N^2 > LM + MN + NL$

   b. a single couple, then $L = M = N, \lambda \neq \frac{1}{2}$.

   State a set of necessary and sufficient conditions for this system to be in equilibrium.

10. ABCD is a squar of \$m, E is on AB such that AE = 3m. The forces λP, μP, υP, 2P, 10P and $2\sqrt{2}$ P Newtons act along the directions BA, BC, CD, AD, DE and DB respectively as indicated by the order of the letters.

   i. When the system is in equilibirium show that $\lambda = \mu = 6$ and $\upsilon = 4$
   ii. If $\upsilon \neq 4$ and $\lambda = \mu = 6$ then show that the system reduces to a single force and find its magnitude direction and line of action.
   iii. If $\upsilon = 2$ and $\lambda = \mu = 6$ then find the magnitude, direction and the line of action of the force that should be added to the system so that the system reduces to a couple of moment 80 Nm.

11. Forces P, 7P, 8P, 7P, 3P newtons act along the sides AB, CB, CD, ED, FE respectively of a regular hexagon ABCDEF of side a meters in the direction indicated by the order of letters. Taking $\underline{i}$ and $\underline{j}$ to be unit vectors in the direction $\overrightarrow{AB}$ and $\overrightarrow{AE}$ respectively. Express each forces interms of $\underline{i}, \underline{j}$ and P.

   Show that the given system is equivalent to a single resultant force $\underline{R} = 2P\left(\underline{i} + \sqrt{3}\underline{j}\right)$ parallel to $\overrightarrow{BC}$.

   What is the magnitude of RP

   Show that the line of action of the resultant passes through the common points of DE and AF (both produced).

   If the system is equivalent to a force $\underline{R}$ acting through the vertex A together with a couple, find the moment of the couple, in magnitude and sense.

12. The coordinates of the points A, B and C with respect to a rectangular cartesian axes Ox and Oy are $\left(\sqrt{3}, 0\right)$, $(0, -1)$ and $\left(2\frac{\sqrt{3}}{3}, 1\right)$ respectively. Forces of magnitude 6P, 4P, 2P and $2\sqrt{3}$ P newtons act along OA, BC, CA and BO respectively in the directions indicated by the order of letters. Find the magnitude and direction of the resultant of these forces. Find the point at which the line of action of the resultant of these forces.

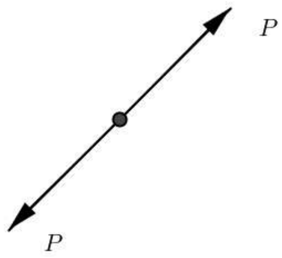
   Find the point at which the line of action of the resultant cuts the y axis.

   Hence find the equation of the line of action of the resultant.

   Another force of magnitude $6\sqrt{3}$ P newtons is introduced to the system along $\overrightarrow{AB}$ show that the system is reduced to a couple of magnitude 10P newton metre.

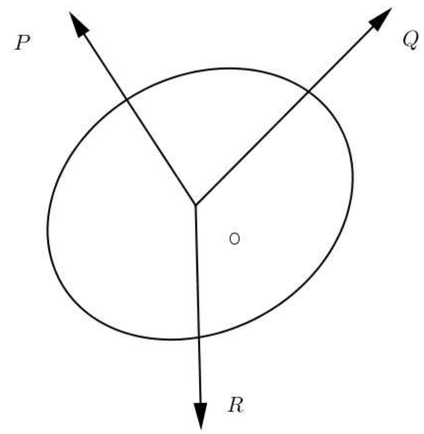
4.4 Equilibrium of a rigid body under the action of coplanar forces

(1) Under the action of two forces



If the two forces are equal in magnitude acting in opposite direction along the same line then the body will be in equilibrium.

(2) Under the action of three Forces



We have two cases to consider.

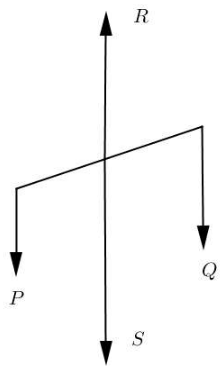
(i) All three forces are not parallel
(ii) They all parallel

In (i) they all should meet at one point and resultant of any two forces should equal and oppsite direction to the third one.

Proof:

Let P,Q,R be the three forces acting on a rigid body and P,Q meet at a point O .Then P,Q have a resultant passing through O. Now we have twi forces, therefore for equilibrium R should pass through O and equal, opposite direction with the resultant of P and Q.

In (ii) all P,Q,R are parallel.



Let P,Q are like parallel forces have a resultant S parallel to P or Q. Now we have two parallel forces S and R. For equilibrium S and R to be equal, dislike parallel and to be in the same line of action , otherwise they have a resultant or forms a couple.

When a rigid body is in equilibrium under the action of three coplanar forces the following results can be used

- i Lami's Theorem
- ii Triangle of Forces
- iii The sum of resolved components along two perpendicular direction are zero

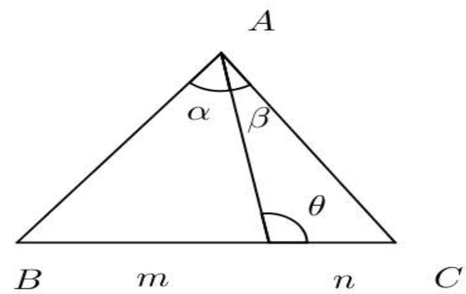
Also the following Trigonometric theorem is useful in dealing with equilibrium problems.

Theorem :

In a triangle ABC let D is a point on BC such that BD : DC = m : n and $B\hat{A}D = \alpha$, $C\hat{A}D = \beta$, $A\hat{D}C = \theta$ then

(i) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

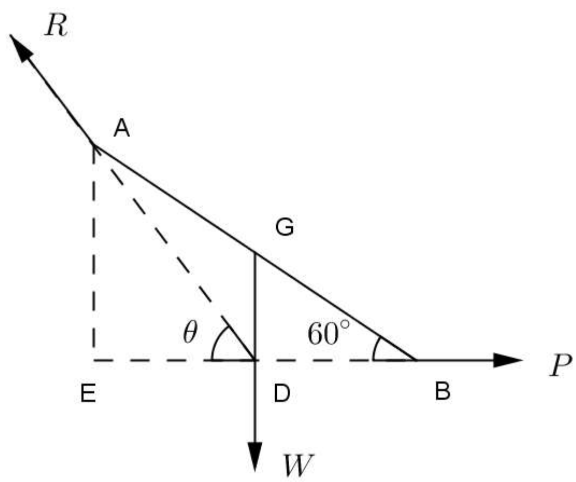
(ii) $(m+n)\cot\theta = n\cot B - m\cot C$



4.5 Worked examples

Example 1:

A heavy uniform rod AB is hinged at A to a fixed point and rests in a position inclined 60 to the horizontal being acted upon by a horizontal force P applied at the lower end B. Find the magnitude of P and the reaction at hinge.



The forces acting are

(i) Weight W of the rod acting vertically through the mid point of the rod.

(ii) The horizontal force P at B.

(iii) The reaction R at hinge A.

The rod is in equilibrium of under the action of three forces, therefore must meet at one point. say D

Let $AB = 2a$, $\angle ADE = \theta$

$AE = 2a\sin 60° = \sqrt{3}a$

Method (i) Using Lami's Theorem

$$\frac{P}{\sin(90+\theta)} = \frac{R}{\sin 90°} = \frac{W}{\sin(180-\theta)}$$

$$\frac{P}{\cos\theta} = R = \frac{W}{\sin\theta}$$

$$P = W\cot\theta$$

$$R = W\cos ec\theta$$

$$P = \frac{W}{2\sqrt{3}} = \frac{\sqrt{3}W}{6}N$$

$$R = W\sqrt{\frac{13}{12}}N$$

$$ED = \frac{1}{2}\times 2a\cos 60° = \frac{a}{2}$$

$$\tan\theta = \frac{AE}{ED} = 2\sqrt{3}$$

$$\cos ec\theta = \sqrt{1+\cot^2\theta}$$

$$\cos ec\theta = \sqrt{1+\frac{1}{12}} = \sqrt{\frac{13}{12}}$$

Method (ii)

In triangle AED, AE is parallel to W, and ED,DA can represent P and R .

ie, ΔAED is the triangle of forces

R $\rightarrow$ DA

W $\rightarrow$ AE

P $\rightarrow$ ED

then $\frac{\text{P}}{\text{ED}} = \frac{\text{W}}{\text{AE}} = \frac{\text{R}}{\text{DA}}$

$$\frac{\text{P}}{\frac{\text{a}}{2}} = \frac{\text{W}}{\sqrt{3}\,\text{a}} = \frac{\text{R}}{\frac{\sqrt{13}\,\text{a}}{2}}$$

$$\text{AD} = \sqrt{3a^2+\frac{a^2}{4}} = \frac{\sqrt{13}}{2}a$$

$\text{P}=\frac{\text{W}}{2\sqrt{3}}$ and $\text{R}=\text{W}\sqrt{\frac{13}{12}}$

Method (iii) Resolving Forces

Resolving horizontally $\rightarrow$

$\text{P - R}\cos\theta = 0$

$\text{P} = \text{R}\cos\theta$

Resolving vertically $\uparrow$

$\text{R}\sin\theta - \text{W} = 0$

$$\text{R} = \frac{\text{W}}{\sin\theta} = \text{W}\sqrt{\frac{13}{12}}N$$

$$\text{P} = \text{W}\cot\theta = \frac{\text{W}}{2\sqrt{3}}N$$

Example 2:

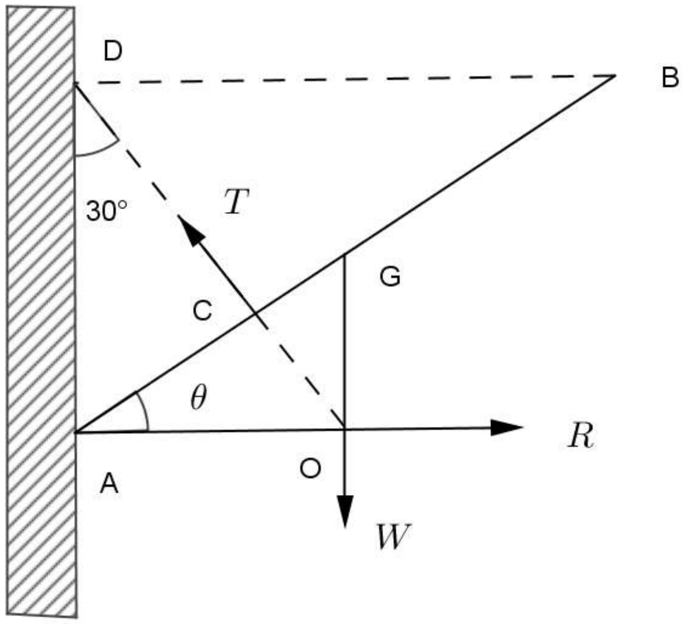
A uniform rod ABC of weight W is supported with B being uppermost, with its end A against a smooth vertical wall AD by means of a string CD, DB being horizontal and CD is inclined to the wall at an angle of 30 . Find

i The tension in the string

ii Reaction of the wall

iii The inclination of the rod

iv Prove that $AC = \frac{1}{3}AB$



The forces acting are ,

i weight W, vertically downward through G, $AG = a$

ii Reaction R at A , horizontal force R

iii Tension in the string T

Since the rod is in equilibrium three forces should meet at one point is O.

Let θ be the inclination of the rod to horizontal

Resolving horizontally $\rightarrow$

$$T\sin 30^\circ - R = 0$$

$$R = \frac{T}{2}$$

Resolving vertically $\uparrow$

$$T\cos 30 - W = 0$$

$$T = \frac{2W}{\sqrt{3}} = \frac{2\sqrt{3}}{3}W \Rightarrow R = \frac{W}{\sqrt{3}} = \frac{\sqrt{3}W}{3}$$

For equilibrium of AB, Taking moment about D

$$R \times AD - W \times AO = 0$$

$$\frac{W}{\sqrt{3}} \times 2a\sin\theta = W \times a\cos\theta \Rightarrow \tan\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

In triangle ACD using sin rule

$$\frac{AC}{\sin 30} = \frac{AD}{\sin(120-\theta)}$$

$$\frac{AC}{\frac{1}{2}} = \frac{2a\sin\theta}{\cos(30-\theta)}$$

$$AC = \frac{a\sin\theta}{\cos 30\cos\theta + \sin 30\sin\theta} = \frac{a}{\cos 30\cot\theta + \sin 30}$$

$$= \frac{2a}{\sqrt{3}\times\frac{2}{\sqrt{3}}+1} = \frac{2a}{3}$$

$$AC = \frac{1}{3}AB$$

Method 2 : By Lami's Theorem

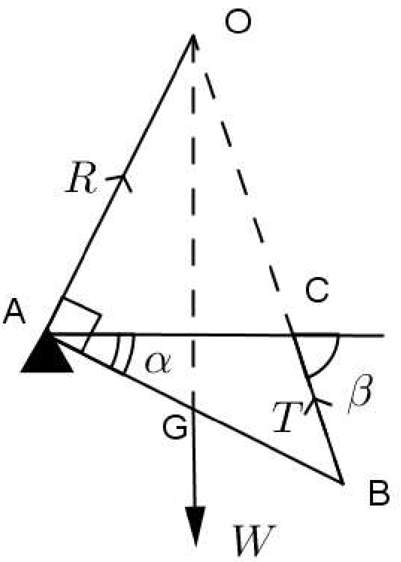
$$\frac{T}{\sin 90} = \frac{W}{\sin 180} = \frac{R}{\sin 150}$$

$$T = \frac{W}{\cos 30} \qquad R = \frac{\cos 60}{\cos 30}$$

$$T = \frac{2W}{\sqrt{3}}N \qquad R = \frac{W}{\sqrt{3}}N$$

Example 3:

A uniform rod AB is in equilibrium at an angle α with the horizontal with its upperend A resting against a smooth peg and its lower end B attached to a light cord, which is pastned to a point C on the same level as A. Prove that the angle β at which the cord is inclined to the horizontal is given by the equation $\tan\beta = 2\tan\alpha + \cot\alpha$ and $AC = \frac{AB\sec\alpha}{1+2\tan^2\alpha}$



The forces acting,

- i Weight W
- ii Tension in the cord T
- iii Reaction at Peg R, perpendicular to the rod

The rod is in equilibrium under the action of thre forces , they meet at point O
In triangle AOC, using cot Rule

$$(AG+GB)\cot\angle OGB = GB\cot 90 - AG\cot\angle ABO$$

$$(1+1)\cot(90+\alpha) = 1\times\cot 90 - 1\times\cot(\beta-\alpha)$$

$$2\tan\alpha = \cot(\beta-\alpha)$$

$$2\tan\alpha = \frac{1+\tan\beta\tan\alpha}{\tan\beta-\tan\alpha}$$

$$\tan\beta(2\tan\alpha-\tan\alpha) = 1+2\tan^2\alpha$$

$$\tan\beta = \frac{1+2\tan^2\alpha}{\tan\alpha}$$

$$\tan\beta = \cot\alpha + 2\tan\alpha$$

$AG:GB = 1:1$

$\angle OBA = \beta-\alpha$

$\angle OAB = 90°$

$\angle OGB = 90° + \alpha$

Using sin Rule, in triangle ABC

$$\frac{AC}{\sin(\beta-\alpha)} = \frac{AB}{\sin(180-\beta)}$$

$$AC = \frac{AB\sin(\beta-\alpha)}{\sin\beta}$$

$$AC = \frac{AB}{\sin\beta}[\sin\beta\cos\alpha - \cos\beta\sin\alpha]$$

$$AC = AB[\cos\alpha - \cot\beta\sin\alpha]$$

$$= AB\left[\cos\alpha - \frac{\tan\alpha}{1+2\tan^2\alpha}\sin\alpha\right]$$

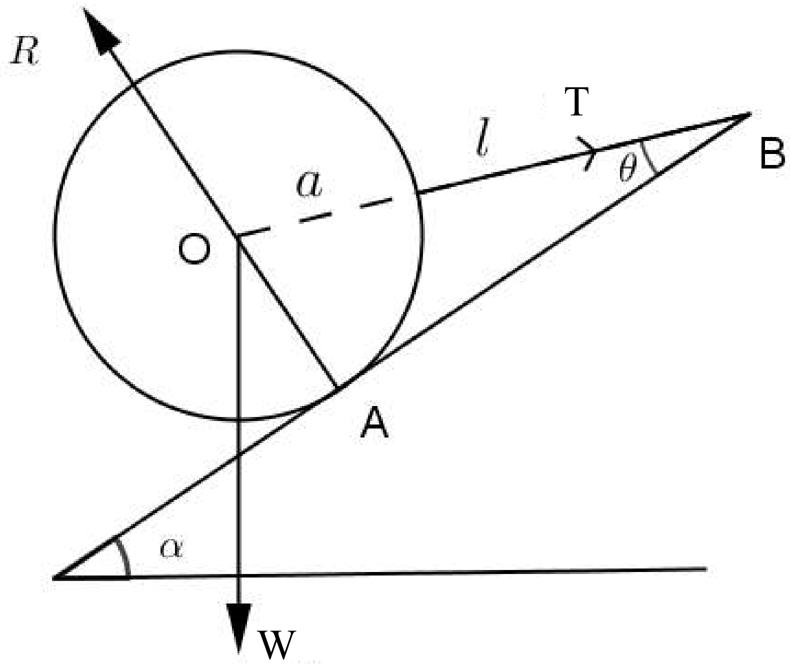
$$= \frac{AB}{1+2\tan^2\alpha}\left[\cos\alpha + \frac{2\sin^2\alpha}{\cos\alpha} - \frac{\sin^2\alpha}{\cos\alpha}\right]$$

$$= \frac{AB}{1+2\tan^2\alpha}\left[\frac{\cos^2\alpha+\sin^2\alpha}{\cos\alpha}\right]$$

$$AC = \frac{AB\sec\alpha}{1+2\tan^2\alpha}$$

Example 4:

A sphere of radius a and weight W rests on a smooth inclined plane supported by a string of length l with one end attached to a point on the surface of the sphere and other end fastened to a point on the plane. If the inclination of the plane to the horizontal is α, prove that the tension in the string is $\dfrac{W(a+l)\sin\alpha}{\sqrt{l^2+2al}}$



The forces acting are

- i Weight W of the sphere, vertically downward through its centre O
- ii Reaction R of the plane perpendicular to the plane, pass through centre O
- iii Tension in the string T

The sphere is in equilibrium under three forces tension in the string should pass through O

In triangle AOB

$$OB = a + l$$

$$OA = a$$

$$AB^2 = (a+l)^2 - a^2 = l^2 + 2al$$

$$AB = \sqrt{l^2+2al}$$

Method 1

Resolving parallel to the plane

O $T\cos\theta - W\cos(90-\alpha) = 0$

$$T = \frac{W\sin\alpha}{\cos\theta}$$

$$= W\sin\alpha.\frac{(a+l)}{\sqrt{l^2+2al}} = \frac{W(a+l)\sin\alpha}{\sqrt{l^2+2al}}$$

$$\cos\theta = \frac{AB}{OB}$$

$$\cos\theta = \frac{\sqrt{l^2+2al}}{a+l}$$

Method 2 : **(Lami's Theorem)**

$$\frac{R}{\sin(90+\alpha-\theta)}=\frac{W}{\sin(90+\theta)}=\frac{T}{\sin\theta}$$

$$T=\frac{W\sin\alpha}{\cos\theta}=\frac{W(a+l)\sin\alpha}{\sqrt{l^2+2al}}$$

$$R=\frac{W}{\cos\theta}.\cos(\alpha-\theta)$$

$$=W(\cos\alpha+\sin\alpha\tan\theta)$$

$$=W\left(\cos\alpha+\sin\alpha.\frac{a}{\sqrt{l^2+2al}}\right)$$

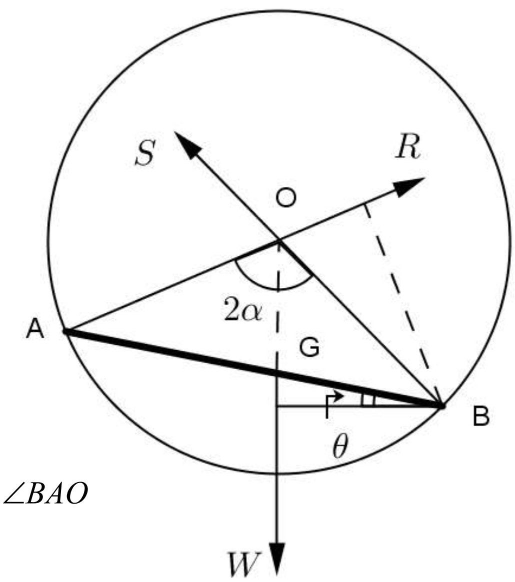
Example 5

Arod of weight W whose centre of gravity divides its length in the ratio 2:1 lies in equilibrium inside a smooth hollow sphere. If the rod subtends an angle 2α at the centre of the sphere and makes angle θ with the horizontal, prove that $\tan\theta=\frac{1}{3}\tan\alpha$. Also find the reactions at the end of the rod interms of W and α

The forces acting are

(i) Weight of the rod W acting through centre O

(ii) Reactions at rhe ends A and B passes through the centreO



$\angle OCA=90-\theta$

$\angle OAB=\angle OBA=90-\alpha$

Using cot rule in triangle AOB

$$(BC+CA)\cot(90-\theta)=CA.\cot\angle ABO-BC.\cot\angle BAO$$

$$3\cot(90-\theta)=2\cot(90-\alpha)-1.\cot(90-\alpha)$$

$$3\tan\theta=\tan\alpha$$

$$\tan\theta=\frac{1}{3}\tan\alpha$$

For equilibrium of AB, taking moment about B

n$R\times 3a\sin(90-\alpha)-Wa\cos\theta=0$

$3R\cos\alpha=W\cos\theta$

$$R = \frac{W\cos\theta}{3\cos\alpha}$$

$$R = \frac{W}{3\cos\alpha}\cdot\frac{3}{\sqrt{9+\tan^2\alpha}}$$

$$R = \frac{W}{\sqrt{9\cos^2\alpha+\sin^2\alpha}} = \frac{W}{\sqrt{8\cos^2\alpha+1}}$$

$$\sec^2\theta = \frac{9+\tan^2\alpha}{9}$$

$$\cos\theta = \frac{3}{\sqrt{9+\tan^2\alpha}}$$

Resolving along the rod

$$S.\cos(90-\alpha) - R.\cos(90-\alpha) - W.\cos(90-\theta) = 0$$

$$S.\sin\alpha - R.\sin\alpha - W\sin\theta = 0$$

$$S.\sin\alpha = R\sin\alpha + W\sin\theta$$

$$= \frac{W}{\cos\alpha}\frac{\sin\alpha}{\sqrt{9+\tan^2\alpha}} + W\frac{\tan\alpha}{\sqrt{9+\tan^2\alpha}}$$

$$= \frac{2W\tan\alpha}{\sqrt{9+\tan^2\alpha}}$$

$$S = \frac{2W}{\cos\alpha\sqrt{9+\tan^2\alpha}}$$

$$= \frac{2W}{\sqrt{8\cos^2\alpha+1}}$$

Example-11:

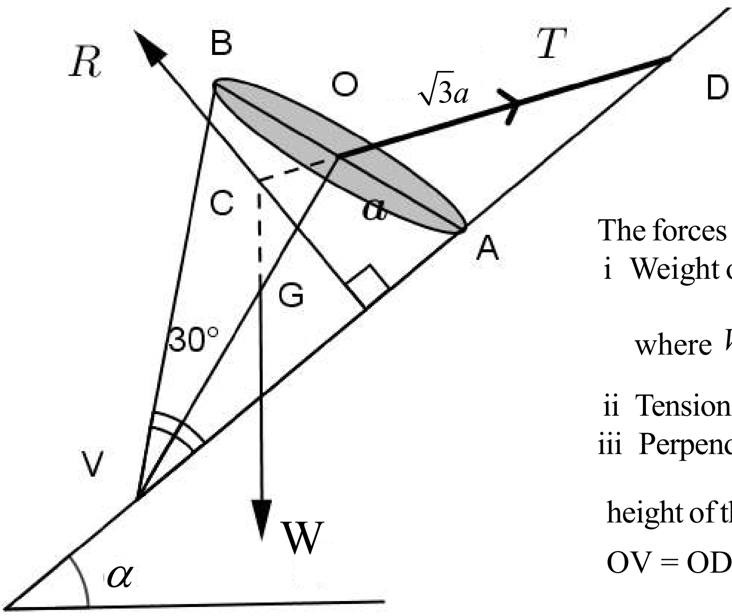
A right circular solid cone of weight W, semivertex angle $30°$, and radius of the base a is placed on a smooth inclined plane which makes an angle α to the horizontal. One end of an inextensible string of the length $\sqrt{3}a$ is attached to the centre of the base of the cone and the other end is connected to the inclined plane. If the system is in equilibrim with curved surface is in contact with the plane

i Show that the tension in the system is $\dfrac{2\sqrt{3}W\sin\alpha}{3}$

ii Find the reaction between the curved surface and the plane

iii Also show that the line of the reaction cuts the symmetric axis of the cone at a distance $\dfrac{3a}{4}\left[\dfrac{3\sqrt{3}\cos\alpha+5\sin\alpha}{3\cos\alpha+\sqrt{3}\sin\alpha}\right]$ from its vertex.

(The centre of gravity of a solid cone of height h is at a distance $\dfrac{3h}{4}$ from the vertex)



The forces acting on the cone are

i Weight of the cone W through G ,

where $VG = \frac{3}{4}a\sqrt{3}$

ii Tension in the string

iii Perpendicular reaction of the smooth plane

height of the cone $= a\cot 30° = a\sqrt{3}$

OV = OD $\Rightarrow \angle ODA = 30°$

For equilibrium, Forces W and T meet at C so reaction R should pass through C

Resolving parallel to the plane

$$\text{O}\ T\cos 30 - W\sin\alpha = 0$$

$$T = \frac{2\sqrt{3}W\sin\alpha}{3}$$

Resolving perpendicular to the plane

$$\text{M}\ R - W\cos\alpha - T\sin 30 = 0$$

$$R = \frac{T}{2} + W\cos\alpha = \frac{W}{3}\left[\sqrt{3}\sin\alpha + 3\cos\alpha\right]$$

Taking moment about V

$$\text{m}R\times x\cos 30 - W.\frac{3}{4}a\sqrt{3}\cos(30+\alpha) - T\sin 30\times 2a\sqrt{3}\frac{\sqrt{3}}{2} = 0$$

$$R.x\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}W}{4}\left[\frac{\sqrt{3}}{2}\cos\alpha - \frac{1}{2}\sin\alpha\right] + \frac{1}{2}\frac{2\sqrt{3}W}{3}\sin\alpha\times 3a$$

$$R.x = \frac{3Wa}{4}\left(\sqrt{3}\cos\alpha - \sin\alpha\right) + 2a\sin\alpha$$

$$= \frac{3W}{4}\left(3\sqrt{3}\cos\alpha - 3\sin\alpha + 8\sin\alpha\right)a$$

$$x = \frac{W}{4}\left(\frac{3\sqrt{3}\cos\alpha + 5\sin\alpha}{3\cos\alpha + \sqrt{3}\sin\alpha}\right)\frac{3a}{W}$$

$$x = \frac{3a}{4}\left(\frac{3\sqrt{3}\cos\alpha + 5\sin\alpha}{3\cos\alpha + \sqrt{3}\sin\alpha}\right)$$

4.6 Equilibrium under the action of more than three forces

Now we have to consider the general case where there are more than three coplanar forces acting on a rigid body. The forces need not meet at one point.

Any system of forces acting in one plane upon a rigid body can be reduced to a single force R or a single couple G
If R=0 its components in any direction is zero.
But $R^2 = X^2 + Y^2$ implies R=0 means X = 0 and Y = 0
ie, The sum of the components in two perpendicular direction each must be zero.
Moment of the couple is the same about any points in its plane we show that if G is zero sum of the moments of the forces about any point in its plane is zero.

Condition for Equilibrium which is sufficient to ensure the equilibrium

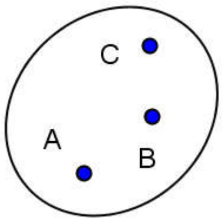
- i The sum of the components of the forces in any two direction must be zero and
- ii The algebraic sum of the moments of the forces about any point in their plane is zero.

Condition (i) ensures that the system does not reduce to a single force and (ii) ensures not reduces to a couple

Another equivalant conditions

The algebraic sum of moments of the force about any three points in its plane not all in a straight line must be zero.

Proof
Total sum of the moments about A or B is zero means there may be resultant and AB its line of action and sum of moment about C is zero.implies there is no such a force.



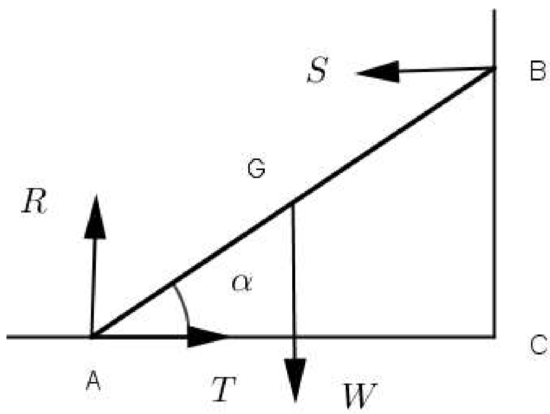
4.7 Worked examples

Example 1

Auniform ladder rests at an angle α to the horizontal with its ends resting on a smooth floor and against a smooth vertical wall. The lower end being joined by a string to the junction of wall and floor. Find the tension of the string and the reaction of the wall and the ground.
Find also the tension of the string when a man of equal weight of the ladder has as centered the ladder three quarters of its length.

Four forces acting on the ladder

- i Weight W
- ii Tension of the string T
- iii Reaction of the floor R
- iv Reaction at wall S



For equilibrium of AB

$\rightarrow \ T - S = 0$

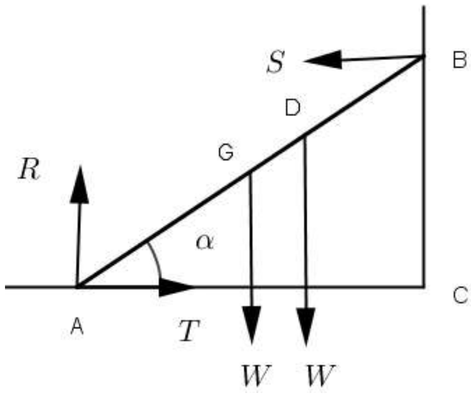
$T = S$................(1)

$\uparrow$ For equilibrium of AB

$R - W = 0$

$R = W$................(2)

Taking moment about A



$$S \times 2a\sin\alpha - W \times a\cos\alpha = 0$$

$$S = \frac{W}{2}\cot\alpha, T = \frac{W}{2}\cot\alpha$$

When the man is on the ladder

$$R - 2W = 0$$

$$R = 2W$$

Moment about B

$$\uparrow \ T \times 2a\sin\alpha - R \times 2a\cos\alpha + W \times a\cos\alpha + W \times \frac{a}{2}\cos\alpha = 0$$

$$2T\sin\alpha = 2 \times 2W\cos\alpha - \frac{3}{2}W\cos\alpha$$

$$T = \frac{5W\cos\alpha}{4\sin\alpha} = \frac{5W}{4}\cot\alpha$$

Example 2

A beam of weight W is divided by its centre of gravity G into two portions AC and BC whose lengths are a and b. The beam rests in a vertical plane on a smooth floor AD and against a smooth vertical wall DB. A string is attached to a hook at D and to the beam at a point P. If T is the tension of the string and θ, ϕ be the inclination of the beam and string to the horizontal respectively.

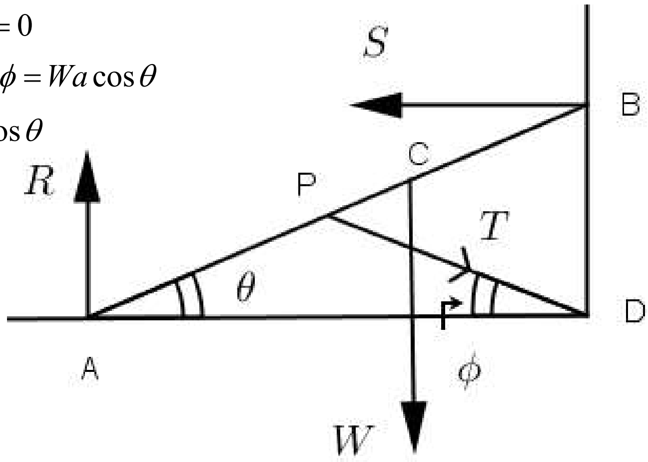
Show that $T = \dfrac{Wa\cos\theta}{(a+b)\sin(\theta - \phi)}$

For equilibrium of AB

$\rightarrow T\cos\phi - S = 0$

$S = T\cos\phi$

Taking moment about A



m $S \times AB \sin\theta - T \times AD \sin\phi - W \times a\cos\theta = 0$

$$T\cos\phi(a+b)\sin\theta - T\times(a+b)\cos\theta\sin\phi = Wa\cos\theta$$

$$T(a+b)[\sin\theta\cos\phi - \cos\theta\sin\phi] = Wa\cos\theta$$

$$T(a+b)\sin(\theta-\phi) = W\times a\cos\theta$$

$$T = \frac{Wa\cos\theta}{(a+b)\sin(\theta-\phi)}$$

Example 3

To the end B of a uniform rod AB of weight W is attached a particle of weight w . The rod and a particle are suspended from a fixed point O by two light strings OA,OB of the same length as the rod. Prove that in equilibrium position, if T_1 and T_2 are the tensions in strings OA,OB then

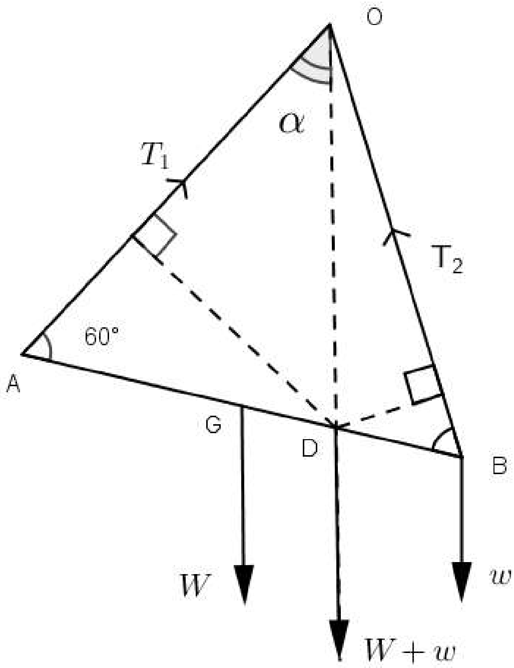
(i) $\dfrac{T_1}{T_2} = \dfrac{W}{W+2w}$

(ii) If α is the angle OA makes with vertical $\tan\alpha = \dfrac{(W+2w)\sqrt{3}}{3W+2w}$

The forces acting are

* weight of the rod W
* weight of the particle w
* Tensions in the strings T_1 and T_2

The resultant of the parallel forces W and w is a like parallel force through D of magnitude W+w where GD:DB=w : W



Let $AB = 2a$ then $GB = a$

$$GD = \frac{w}{W+w}a$$

$$AD = a + \frac{wa}{W+w} = \left(\frac{W+2w}{W+w}\right)a \quad \text{and}$$

$$DB = a - \frac{wa}{W+w} = \frac{Wa}{W+w}$$

Now we have three forces in equilibrium taking moment about D

n $T_1 \times AD\sin 60 - T_2 \times DB\sin 60 = 0$

$$\frac{T_1}{T_2} = \frac{DB}{AD} = \frac{Wa}{W+w}\times\frac{W+w}{(W+2w)a} = \frac{W}{W+2w}$$

Using sin rule in triangle OAD

$$\frac{AD}{\sin\alpha}=\frac{OA}{\sin\left[180-(60+\alpha)\right]}$$

$$\frac{AD}{OA}=\frac{\sin\alpha}{\sin(60+\alpha)}$$

$$\frac{W+2w}{W+w}\times\frac{a}{2a}=\frac{\sin\alpha}{\frac{\sqrt{3}}{2}\cos\alpha+\frac{1}{2}\sin\alpha}$$

$$\frac{W+2w}{2(W+w)}=\frac{1}{\frac{\sqrt{3}}{2}\cot\alpha+\frac{1}{2}}=\frac{2}{\sqrt{3}\cot\alpha+1}$$

$$\frac{\sqrt{3}\cot\alpha+1}{2}=\frac{2(W+w)}{W+2w}$$

$$\sqrt{3}\cot\alpha=\frac{4W+4w}{W+2w}-1=\frac{3W+2w}{W+2w}$$

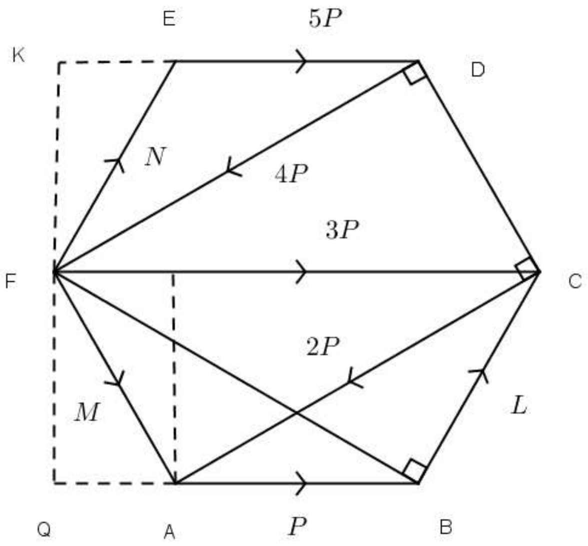
$$\cot\alpha=\left(\frac{3W+2w}{W+2w}\right)\frac{1}{\sqrt{3}}$$

$$\tan\alpha=\left(\frac{W+2w}{3W+2w}\right)\sqrt{3}$$

Example 4

The Points A,B,C,D,E,F are the vertices of a regular hexagon ABCDEF of side $2a$ makes taken in anticlockwise sense. Forces of magnitudes P,2P,3P,4P,5P,L,M,N newtons act along AB,CA,FC,DF,ED,BC,FA and FE respectively in the direction indicated by the order of the letters. If the system is in equilibrium find L,M,N interms of P.

Given that forces are in equilibrium their sum of moments about any point is zero.



$FB \perp BC$

$FD \perp DC$

Taking moment about F

m $L \times FB - 5P \times FK + P \times FQ - 2P \times FA = 0$

$$L \times 4a\cos 30 - 5P \times 2a\sin 60 + P \times 2a\sin 60 - 2P \times 2a = 0$$

$$4L \times \frac{\sqrt{3}}{2} - 10P \times \frac{\sqrt{3}}{2} + 2P \times \frac{\sqrt{3}}{2} - 4Pa = 0$$

$$2\sqrt{3}L - 4\sqrt{3}P - 4P = 0$$

$$L = \frac{4P + 4\sqrt{3}P}{2\sqrt{3}}$$

$$= 2P\left(1 + \frac{1}{\sqrt{3}}\right)N$$

Taking moment about A

m $L \times 2a\cos 30 - 5P \times 4a\cos 30 - N \times 2a\cos 30 - 4P \times 2a - 3P \times 2a\cos 30 = 0$

$$2L \times \frac{\sqrt{3}}{2} - 2N \times \frac{\sqrt{3}}{2} - 26P \times \frac{\sqrt{3}}{2} + 8P = 0$$

$$L - N - 13P + \frac{8P}{\sqrt{3}} = 0$$

$$N = L - 13P + \frac{8P}{\sqrt{3}}$$

$$= 2P + \frac{2P}{\sqrt{3}} + \frac{8P}{\sqrt{3}} - 13P$$

$$N = \left(\frac{10}{\sqrt{3}} - 11\right)P$$

Resolving parallel to AB

$$\rightarrow L\cos 60 + M\cos 60 + N\cos 60 + 5P + 3P + P - 4P\cos 30 - 2P\cos 30 = 0$$

$$\frac{L}{2} + \frac{M}{2} + \frac{N}{2} + 9P - 4P\frac{\sqrt{3}}{2} - 2P\frac{\sqrt{3}}{2} = 0$$

$$L + M + N = 6\sqrt{3}P - 18P$$

$$M = 6\sqrt{3}P - 18P - \left(\frac{12P}{\sqrt{3}} - 9P\right)$$

$$= 2\sqrt{3}P - 9P$$

$$= \left(2\sqrt{3} - 9\right)P$$

Therefore

$$L = \left(1 + \frac{1}{\sqrt{3}}\right)PN$$

$$M = \left(2\sqrt{3} - 9\right)PN$$

$$N = \left(\frac{10}{\sqrt{3}} - 11\right)PN$$

4.8 Exercises

(1) A uniform bar AB of weight $2w$ and length l is free to turn about a smooth hinge at its upperend A, and a horizontal force is applied to the otherend B so that the bar is in equilibrium with B is at a distance a from the vertical through A. Prove that the reaction at the hinge is

equal to $w\left[\dfrac{4l^2-3a^2}{l^2-a^2}\right]^{\frac{1}{2}}$

(2) A uniform rod, of length a hangs against a smooth vertical wall being supported by means of a string of length l, tied to one end of the rod, the other end of the string being attached to a point in the wall. Show that the rod can inclined to t he wall at an angle θ given by

$\cos^2\theta=\dfrac{l^2-a^2}{3a^2}$

What are the limits of the ratio of $a:l$ for which equilibrium is possible.

(3) A sphere of radius r and weight W rests against a smooth vertical wall , in which is attached a string of length l fastened to a point on its surface. Show that the tension in the string is

$\dfrac{W(l+r)}{\sqrt{l^2+2lr}}$

Also find the reaction between wall and sphere.

(4) A solid cone of height h and semi vertical angle α , is placed with its base against a smooth vertical wall is supported by a string attached to the vertex and to a point on the wall. Show that the greatest possible length of the string is $h\sqrt{1+\dfrac{16}{9}\tan^2\alpha}$

(5) A triangular lamina ABC is suspended from a point O by light string fastened to points A and B and hangs so that BC is vertical. Prove that if α and β be the angles which strings AO and BO makes with vertical then $2\cot\alpha-\cot\beta=3\cot\beta$

(6) A uniform rectangular lamina board rests vertically in equilibrium with its side $2a$ and $2b$ on two smooth pegs in the same horizontal line at a distance c apart. Prove that the side of length $2a$ makes with horizontal an angle θ given by $c\cos 2\theta=a\cos\theta-b\sin\theta$
Deduce that a square of side $2a$ will rest on the smooth pegs when the inclination of the side to the horizontal is as $\dfrac{1}{2}\sin^{-1}\left(\dfrac{a^2-c^2}{c^2}\right)$

(7) A uniform rod of weight W rests with its ends contact with two smooth planes inclined at an angle and β respectively to the horizontal and intesecting in a horizontal line. If θ be the inclination of the rod to the vertical show that $2\cot\theta = \cot\beta - \cot\alpha$ also find the reaction at the ends.

(8) A smooth peg is fixed at a point P at a distance a from a smooth verical wall. A uniform rod AB of length $6a$ and weight W is in equilibrium resting on the peg with the end A is in contact with the wall. Taking θ be the angle made by the rod AB with the horizontal draw a triangle of force, respresenting forces acting on the rod. Find the reaction at P interms of W and θ, show that $3\cos^3\theta = 1$

(9) A thin rod of length a is in equilibrium with its ends resting on the inner smooth surface of a smooth circular hoop of radius a, fixed its plane vertical. If the centre of gravity divides its length in the ratio 3:4. Prove that the inclination of the rod to the vertical is $\tan^{-1}\left(\frac{7}{\sqrt{3}}\right)$ Determine the ratio of the reaction on the lower end of the rod to that on upper end.

(10) Two uniform smooth spheres of radius a and weight W lie at rest touching each other inside a fixed smooth hemispherical bowl of radius $b\ (> 2a)$. Draw in seperate diagrams, a triangle of forces representing forces acting on the spheres and show that the reaction between the two spheres is $\frac{Wa}{\sqrt{b(b-2a)}}$

(11) One end of a uniform beam of weight W is placed on a smooth horizontal plane, the other end to which a string is fastened, rests against another smooth inclined plane, inclined at an angle a to the horizontal. The string passes over a pully at the top of the inclined plane, hangs vertically and supports a weight P. Show that in equilibrium $2P = W\sin\alpha$

(12) A rod is movable in a vertical plane about a hinge at one end and at the end is fastened a weight equal to half the weight of the rod . This end is fastened by a string of length l to a point at a height c vertically above the hinge. Show that the tension in the string is $\frac{lW}{c}$ where W is the weight of the rod.

(13) ABCDEF is a regular hexagon. Five forces each equal to P act along AE,ED,DC,CB,BA. Five forces each equal to Q act along AC,CE,EB,BD,DA respectively indicated by the order of letters. Prove that the ten forces will be in equilibrium if P and Q are in a certain ratio and find the ratio?

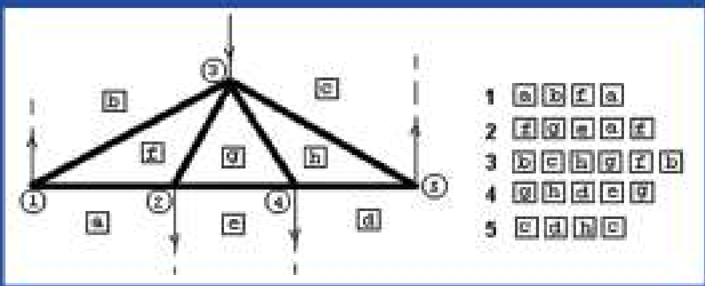
G.C.E. Advanced Level

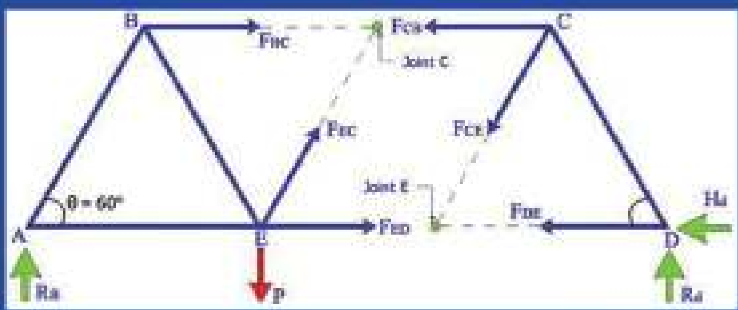
Combined Mathematics

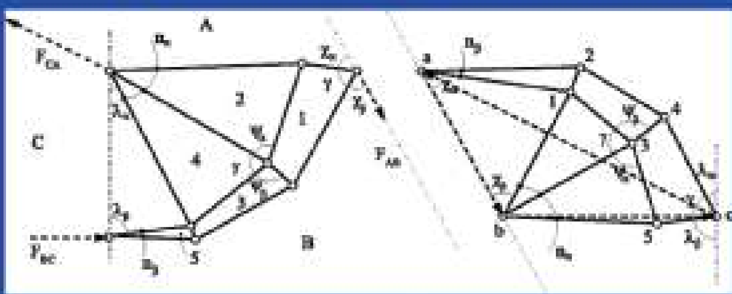
STATICS – II

Additional Reading Book

(Prepared According to the New syllabus Implemented From 2017)







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STATICS - II

Additional Reading Book

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Message from the Director General

Department of Mathematics of National Institute of Education time to time implements many different activities to develop the mathematics education. The publication of this book is a mile stone which was written in the name of **"Statics - Part I, Statics - Part II".**

After learning of grade 12 and 13 syllabus, teachers should have prepared the students for the General Certificate of Education (Advanced Level) which is the main purpose of them. It has not enough appropriate teaching - learning tools for the proper utilization. It is well known to all, most of the instruments available in the market are not appropriate for the use and it has not enough quality in the questions. Therefore **"Statics - Part I, Statics - Part II".** book was prepared by the Department of Mathematics of National Institute of Education which was to change of the situation and to ameliorate the students for the examination. According to the syllabus the book is prepared for the reference and valuable book for reading. Worked examples are included which will be helpful to the teachers and the students.

I kindly request the teachers and the students to utilize this book for the mathematics subjects' to enhance the teaching and learning process effectively. My gratitude goes to Aus Aid project for sponsoring and immense contribution of the internal and external resource persons from the Department of Mathemetics for toil hard for the book of **"Statics - Part I, Statics - Part II".**

Dr. (Mrs). T. A. R. J. Gunasekara
Director General
National Institute of Education.

Message from the Director

Mathematics holds a special place among the G.C.E. (A/L) public examination prefer to the mathematical subject area. The footprints of the past history record that the country's as well as the world's inventor's spring from the mathematical stream.

The aim and objectives of designing the syllabus for the mathematics stream is to prepare the students to become experts in the Mathematical, Scientific and Technological world.

From 2017 the Combined Mathematics syllabus has been revised and implemented. To make the teaching - learning of these subjects easy, the Department of Mathemactics of National Institute of Education has prepared **Statics - Part 1 and Part 11** as the supplementary reading books. There is no doubt that the exercises in these books will measure their achievement level and will help the students to prepare themselves for the examination. By practicing the questions in these books the students will get the experience of the methods of answering the questions. Through the practice of these questions, the students will develop their talent, ability, skills and knowledge. The teachers who are experts in the subject matter and the scholars who design the syllabus, pooled their resources to prepare these supplementary reading books. While preparing these books, much care has been taken that the students will be guided to focus their attention from different angles and develop their knowledge. Besides, the books will help the students for self-learning.

I sincerely thank the Director General for the guidance and support extented and the resource personnel for the immense contribution. I will deeply appreciate any feedback that will shape the reprint of the books.

Mr. K. R. Pathmasiri
Director
Department of Mathematics

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Preface

This book is being prepared for the students of Combined Mathematics G.C.E.A/L to get familiar with the subject area of Statics. It is a supplementary book meant for the students to get practice in answering the questions for self learning. The teachers and the students are kindly invited to understand, it is not a bunch of model questions but a supplementary to encourage the students towards self learning and to help the students who have missed any area in the subject matter to rectify them.

The students are called upon to pay attention that after answering the questions in worked examples by themselves, they can compare their answers with the answers given in the book. But it is not necessary that all the steps have taken to arrive at the answers should tally with the steps mentioned in the book's answers given in this book are only a guide.

Statics Part 11 is released in support of the revised syllabus - 2017. The book targets the students who will sit for the GCE A/L examination – 2019 onwards. The Department of Mathematics of National Institute of Education already released **Practice Questions and Answers** book and book of **'Statics – I ',** it is being proceeded by the **" Statics II".** There are other two books soon be released with the questions taken Unit wise **"Questions bank".**

We shall deeply appreciate your feedback that will contribute to the reprint of this book.

Mr.S.Rajendram
Project Leader
Grade 12, 13 Maths
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Content

5.0 Jointed Rods

In previous chapters 4.1 and 4.2 we considered the action of coplanar forces acting on a single rigid body In this chapter we shall consider the action of coplanar forces acting on two or more rigid bodies, specially on number of heavy rods jointed together to form a frame.

We will consider the equilibirium of rods under the action of their weight, any external forces applied and forces exerted on their ends by hings (joints)

5.1 Types of simple joints

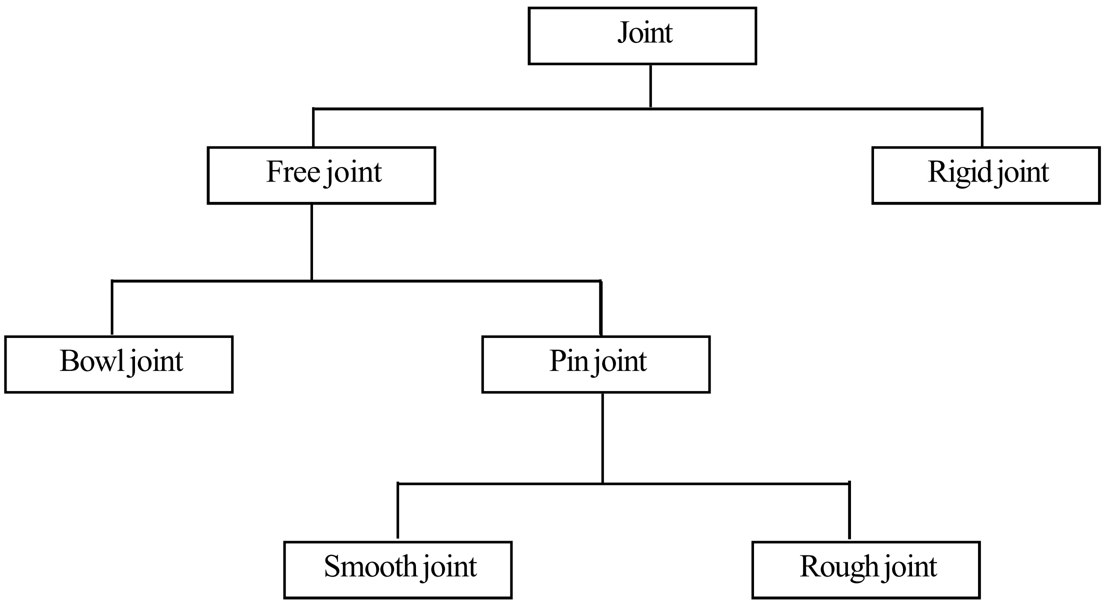
(i) Rigid Joint

When two rods are jointed together such a way that they cannot be seperated or turn about one another at the joint, the joint is rigid joint.

(ii) Pin Joint

When two rods are jointed by a light pin such a way that they can turn at the joint, the joint is pin joint if they can turn freely at the joint, the joint is a smooth pin joint and, if free turn is not possible the joint is a rough joint.

We shall consider the frames with smooth pin joints in this chapter.

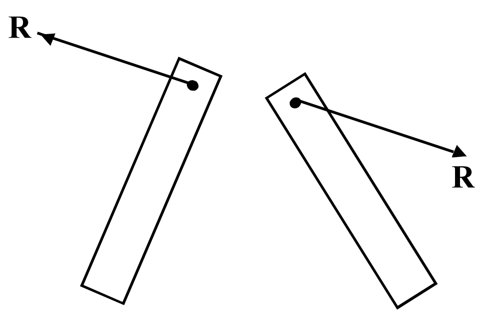


5.2 Rigid joint

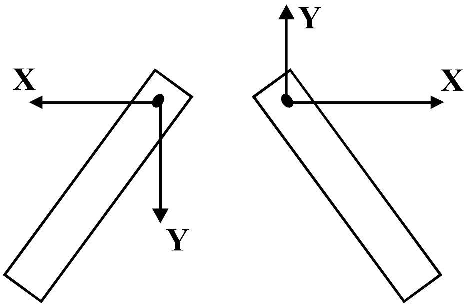
If the shape of a body obtained by joining two or more bodies together cannot be changed by external forces then the joint is said to be a rigid joint.

Force at a smooth joint (Pin joint)

The joints are shown seperately to show the reaction at the joint. The reaction at the joints will be equal and opposite. To find R easily the components of R are shown as follows.



X, Y are the horizontal and vertical components of R. R is the resultant of X and Y, and passes through the pin joint.

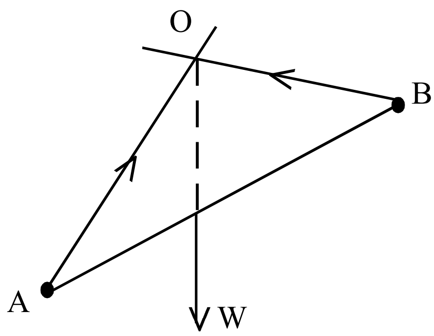


The light pin is assumed to be a small smooth pin of circular rim, joins the rod by passing through the rods. As the pin is smooth the reaction on contact is perpendicular to it and for rods. Since the pin is in equilibirium under these two forces, they are equal opposite in direction and have the sameline of action. Therefore the reaction on each rod is equal and opposite and have the same line of action an each joints.

For conveneience we resolve the reaction into two perpendicular components when we need of it.

Note :

When a heavy rod is joined at its ends to another rod, the reaction by joints on the rod cannot lie along the rod, since the rod is acted by three forces.



For equilibrium forces should meet at one point O which cannot lie on AB.

If the rod is light, it is acted by the two reactions only, so that they always lie along the rod to balance each other.

When a framework is symmetric about an axis identical set of forces will act on both sides.

Instructions to solve problems

(i) Correct diagram has to be drawn with geometrical data.

(ii) Forces should be marked correctly.

(iii) Necessary equations should be obtained to find the unknown forces.

(iv) To find the reactions at a joint the force at the joint should be divided into two components and to be marked.

(If there is axis of symmetry, it should be stated and the results can be used)

Note :

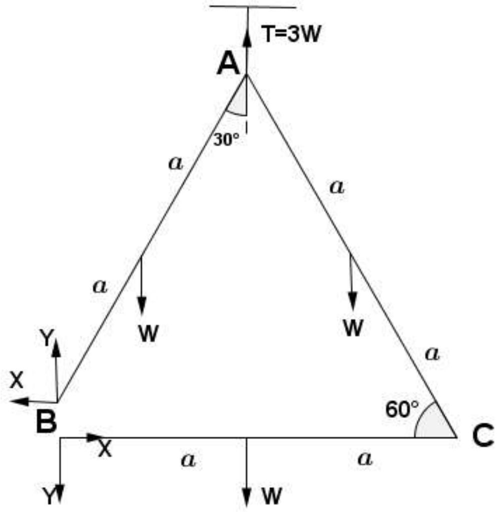
A framework must be rigid. To make a framework of n jonts ($n > 3$) to be rigid it is necessary to have ($2n-3$) rods.

A framework with more than ($2n-3$) rods will make the framework over rigid.

5.3 Worked examples

Example 1

Three uniform equal rods of length $2a$ and weight W are freely jointed at their end points and the frame ABC is suspended from the joint A . Find the magnitude and direction of the reaction at B on AB.



Consider the equilibrium of BC

Taking moments about C for BC

$$C\circlearrowleft \quad W.a + Y.2a = 0$$

$$2Y + W = 0 \qquad ; Y = -\frac{W}{2}$$

Consider the equilibrium of AB.

Taking momements about A for AB

$$A\circlearrowleft \quad Y(2a\sin 30°) + X(2a\cos 30°) - W(a\sin 30°) = 0$$

$$2Y + 2X\cot 30° = W$$

$$2Y + 2\sqrt{3}X = W$$

$$-W + 2\sqrt{3}X = W \qquad ; X = \frac{W}{\sqrt{3}}$$

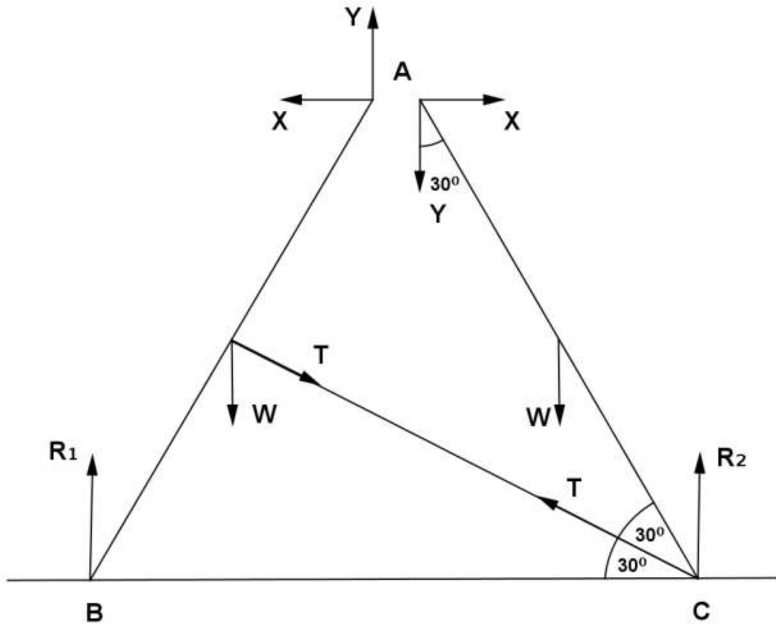
$$R = \sqrt{X^2 + Y^2} = \sqrt{\frac{W^2}{3} + \frac{W^2}{4}} = \sqrt{\frac{7}{12}}W$$

$$\tan\theta = \frac{Y}{X} = \frac{\sqrt{3}}{2}; \qquad \therefore\ \theta = \tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

Magnitude of the reaction at B $=\sqrt{\frac{7}{12}}$ W ; R makes an angle θ with CB where $\theta = \tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Example 2

Two uniform rods AB, AC each of length $2a$ and weight W are smoothly jointed at A. The rods are in equilibrium in a vertical plane with B and C lying on a smooth horizontal plane and C is connected to the midpoint of AB by an inextensible string and $B\hat{A}C = 60°$. Find the tension in the string and the reaction at A.



$AB = AC\ ;\ B\hat{A}C = 60°$

Therefore ABC is an equilateral triangle.

For equilibrium of AB and AC,

Resolving vertically

$\uparrow\ R_1 + R_2 - 2W = 0\ ;\ R_1 + R_2 = 2W$ .... ①

Taking moment about C

Cm $-R_1 . 4a \cos 60° + W . a \cos 60° + W . 3a \cos 60° = 0$ .... ②

$R_1 = W$ and $R_2 = W$

For equilibrium of AC,

Am $-W.a \cos 60° - T . a + R_2 . 2a \cos 60° = 0$ .... ③

$-\dfrac{W}{2} - T + W = 0; \quad T = \dfrac{W}{2}$

For equilibrium of AC, moment about A

Resolving horizontally,

$\rightarrow X - T \cos 30° = 0\ ;\ X = T \cos 30° = \dfrac{\sqrt{3}W}{4}$

Resolving vertically,

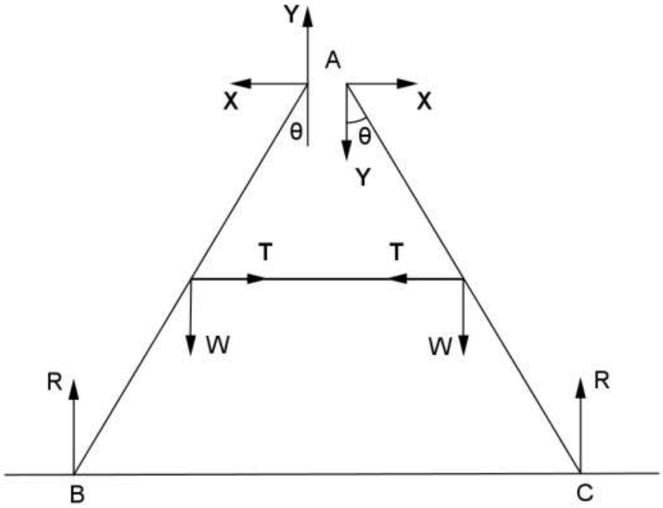
$\uparrow R_2 - Y - W + T \sin 30° = 0$

$Y = R_2 - W + \dfrac{T}{2} = \dfrac{W}{4}$

Hence reaction at A is $\sqrt{X^2 + Y^2} = \sqrt{\dfrac{3W^2}{16} + \dfrac{W^2}{16}} = \dfrac{W}{2}$

Example 3

Two uniform equal rods AB, AC each of weight W are smoothly jointed at A. The ends B and C rest on a horizontal smooth plane and the frame ABC is kept in a vertical plane. The equilibrium is maintained by connecting midpoints of AB and AC by an inextensible string. If $B\hat{A}C = 2\theta$, find the tension in the string and the magnitude of the reaction at A on AB.



Let AB = AC = $2a$

For the equilibrium of AB and AC,

Resolving vertically,

$\uparrow 2R - 2W = 0$

$R = W$ ....①

For equilibrium of AB,

Resolving vertically,

$\uparrow R + Y - W = 0$

$W + Y - W = 0 ; Y = 0$ ....②

Resolving horizontally,

$\rightarrow T - X = 0 ; \; T = X$ .... ③

Taking moment about A for equilibrium of AB,

Am $\quad T.a\cos\theta + W.a\sin\theta - R.2a\sin\theta = 0$

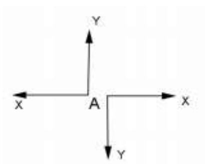
$$T = \frac{(2W - W)\sin\theta}{\cos\theta} = W\tan\theta \text{ ....④}$$

Reaction at A is $W\tan\theta$

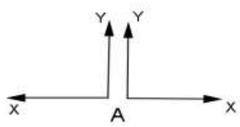
Note :

In the above example the system is symmetrical about the vertical axis through A

Now the reaction at A is given by



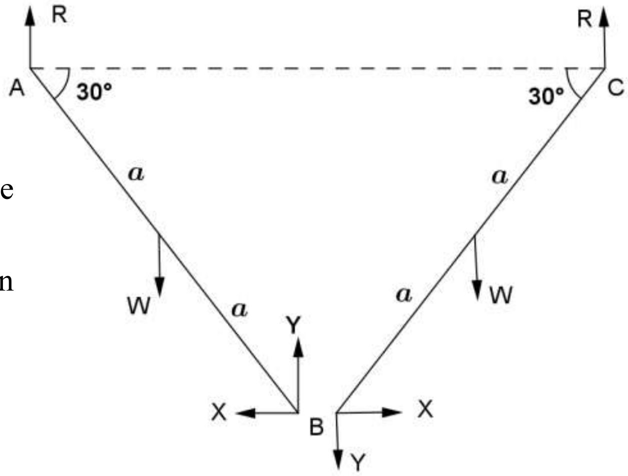
Since the system is symmetri about the vertical axis through A, the forces should be as given below.



Hence Y = 0

Example 4

AB, BC are two uniform rods each of lengh $2a$ and weight W, smoothly hinged at B, and the frame ABC is suspended from the points A and C at the same horizontal level. The systerm is in a vertical plane and each rod makes 30° with the horizontal. Find the reaction at the joint B.



The system is symmetrical about the vertical line through B.

Therefore the vertical component (Y) of the reaction at B is zero (Y=0)

For the equilibrium of AB

By taking moments about A

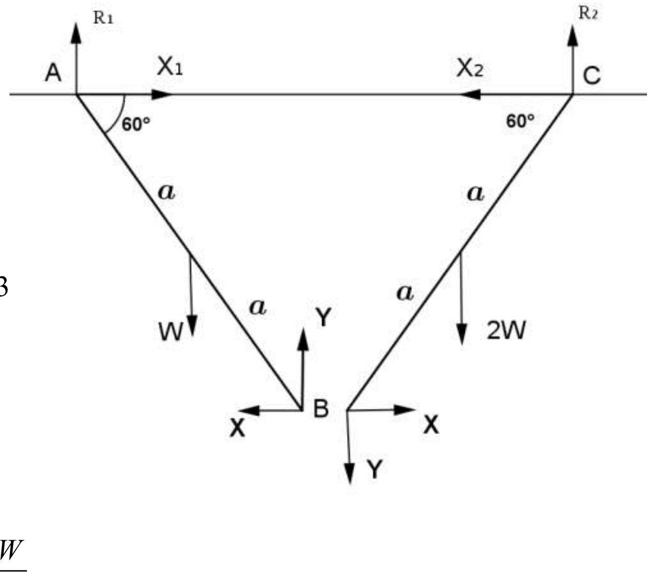
Am $\quad -X \,.\, 2a \sin 30° + Y \,.\, 2a \cos 30° - Wa \cos 30° = 0$

$$-X \,.\, 2a \sin 30° = W.a \cos 30°$$

$$X = -\frac{\sqrt{3}W}{2}$$

Example 5

AB, BC are two equal uniform rods each of length $2a$ and weight W and $2W$ respectively. The rods are smoothly jointed at B and the frame ABC is suspended from A and C at the same horizontal level. The system is in the vertical plane and each rod makes 60° with the horizontal. Find the magnitude and the direction of the reaction at the joint B on AB.



For equilibrium of the system

Resolving horizontally,

$$\rightarrow X_1 - X_2 = 0 \;;\; X_1 = X_2$$

Resolving vertically

$$\uparrow R_1 + R_2 - 3W = 0 \quad ; \quad R_1 + R_2 = 3$$

For AB and AC moment about A

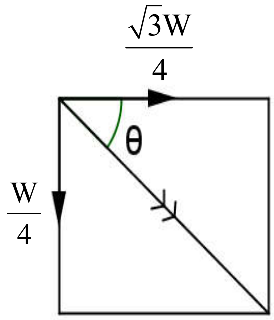
Am $\quad R_2.2a - W.\frac{a}{2} - 2W.\frac{3a}{2} = 0$

$$2R_2 = \frac{7W}{2} \;;\; R_2 = \frac{7W}{4} \text{ and } R_1 = \frac{5W}{4}$$

For equilibrium of BC,

Resolving vertically $\uparrow\; R_2 - 2W - Y = 0 \;;\; Y = R_2 - 2W = \frac{7W}{4} - 2W = -\frac{W}{4}$

For equilibrium of BC



Cm $\quad X\,.\,2a\sin 60^\circ + Y\,.\,2a\cos 60^\circ + 2W\,.\,a\cos 60^\circ = 0$

$$X.\,2a\sin 60^\circ - \frac{W}{4}.\,2a\cos 60^\circ + 2W.\,a\cos 60^\circ = 0$$

$$X = -\frac{\sqrt{3}W}{4}$$

$$R = \sqrt{\frac{3W^2}{16} + \frac{W^2}{16}}$$

$$R = \frac{W}{2}$$

$$\tan\theta = \frac{\frac{W}{4}}{\frac{\sqrt{3}W}{4}} = \frac{1}{\sqrt{3}}$$

$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{\pi}{6}$$

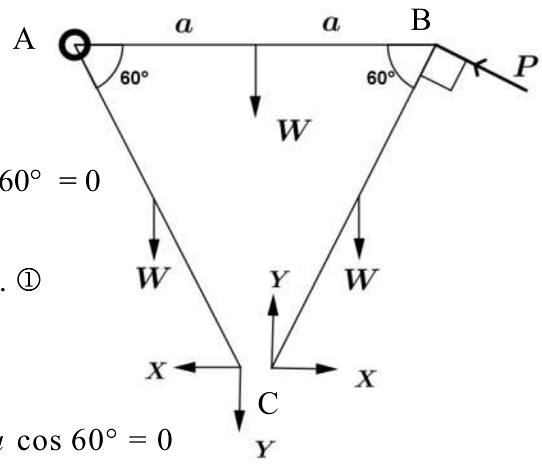
Example 6

Three uniform equal rods AB, BC, AC each of length $2a$ and weight W are smoothly jointed at their ends to form an equilateral triangle. The frame is freely hinged at A in a vertical plane. The triangle is kept in equilibrium with AB as horizontal and C is below AB by a force at B perpendicular to BC by a force P at B perpendicular to BC. AB is horizontal and C is below AB. Find the value of P. Also find the horizontal and vertical components of the horizontal and vertical components of the reaction at C.

By taking moments about A for the system

Am $\quad -W\,.a\cos 60^\circ - W.a - W\,(2a - a\cos 60^\circ) + P.\,2a\cos 60^\circ = 0$

$$P = 3W$$



By taking moment about A for equilibrium of AC,

Am $\quad X.\,2a\sin 60^\circ + Y.\,2a\cos 60^\circ + W.\,a\cos 60^\circ = 0$

$$\Rightarrow X + \frac{Y}{\sqrt{3}} = -\frac{W}{2\sqrt{3}} \quad \text{..............................} \; ①$$

By taking moments about B for equilibrium of BC,

Bm $\quad X\,.\,2a\sin 60^\circ - Y\,.\,2a\cos 60^\circ + W\,.\,a\cos 60^\circ = 0$

$$\Rightarrow X - \frac{Y}{\sqrt{3}} = -\frac{W}{2\sqrt{3}} \quad \text{..............................} \; ②$$

From ① and ②

$$Y = 0$$

$$X = -\frac{W}{2\sqrt{3}}$$

Reaction at C is $\frac{W}{2\sqrt{3}}$

Example 7

Two uniform rods AB and BC each of length $2a$ and weights 2W, W respectively are smoothy hinged at B. The mid points of the rods are connected by a light in inelastic string. The system in a vertical plane with other ends A and C lie on a smooth horizontal table. If $A\hat{B}C = 2\theta$ show that the tension in the string is $\frac{3W}{2}\tan\theta$. Find the magnitude and direction of the reaction at B.

For the equilibrium of the system,

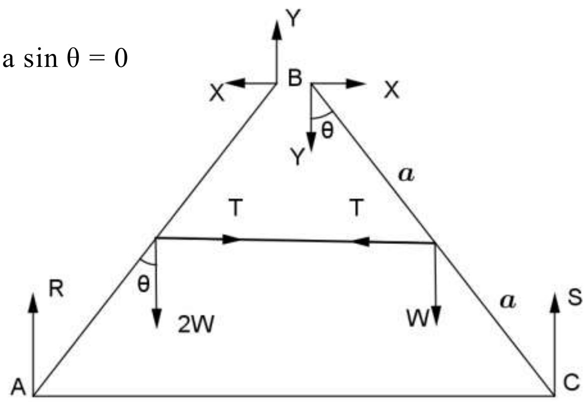
By taking moments about C

Cm $\quad W.\, a \sin\theta + 2W.\, 3a \sin\theta - R.\, 4a \sin\theta = 0$

$$R = \frac{7W}{4}$$

For equilibrium of AB , taking moment about B

Bm $\quad T.\, a\cos\theta + 2W.\, a\sin\theta - R.\, 2a\sin\theta = 0$



$$T = -2W\tan\theta + 2R.\tan\theta$$

$$T = -2W\tan\theta + \frac{7W}{2}\tan\theta$$

$$T = \frac{3W}{2}\tan\theta$$

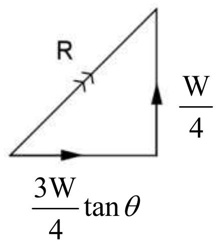
For equilibrium of AB

Resolving horizontally,

$$\rightarrow\ T - X = 0 ;\ X = T = \frac{3W}{2}\tan\theta$$

Resolving vertically,

$$\uparrow\ Y + R - 2W = 0$$



$$Y = 2W - \frac{7W}{4} = \frac{W}{4}$$

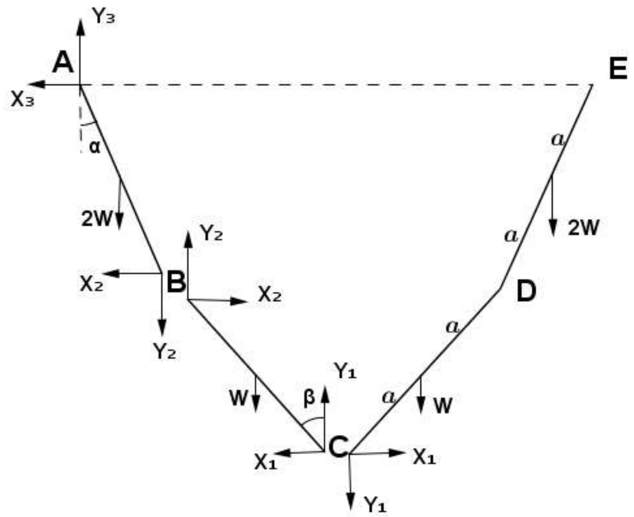
$$R = \sqrt{X^2 + Y^2}$$

$$= \sqrt{\frac{9W^2}{4}\tan^2\theta + \frac{W^2}{16}}$$

$$= \frac{W}{4}\sqrt{1 + 36\tan^2\theta}$$

Example 8

AB, BC, CD, DE are four uniform equal rods of length $2a$, smoothly jointed at B, C and D. The weights of AB, DE are $2W$ each and the weights of BC, CD are W each. The system is suspended from A and E at the same horizontal level. AB and BC make α, β with the vertical respectively. Show that the reaction at C is horizontal and the magnitude is $\frac{W}{2}\tan\beta$. Show also that $\tan\beta = 4\ \tan\alpha$.



The system is symmetrical about the vertical axis through C

Therefore the vertical component of the reaction at C is zero. $Y_1 = 0$

For the equilibrium of BC

Resolving the forces horizontally

$$\rightarrow X_1 - X_2 = 0$$

$$X_1 = X_2$$

Resolving the forces vertically

$$\uparrow Y_1 + Y_2 - W = 0$$

$$Y_2 = W$$

moment about B

B↻ $\quad -X_1 \, . \, 2a\cos\beta - W \, . \, a\sin\beta = 0$

$$X_1 = -\frac{W}{2}\tan\beta$$

For equilibrium of AB,

A↻ $\quad -X_2 \, . \, 2a\cos\alpha + 2W \, . \, a\sin\alpha + Y_2 . \ \ 2a\sin\alpha = 0$

$$X_2 = -2W\ \tan\alpha$$

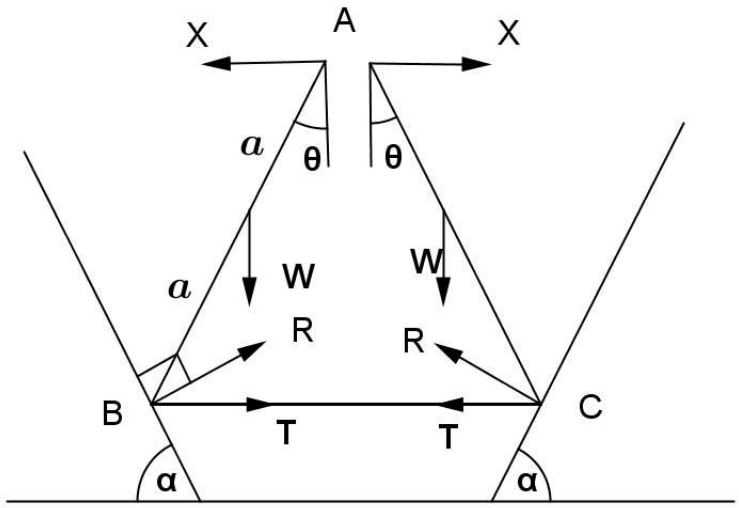
$$X_1 = X_2$$

$$\frac{W}{2}\tan\beta = 2W\ \tan\alpha$$

$$\tan\beta = 4\tan\alpha$$

Example 9

Two equal uniform rods AB and AC each of weight W are freely jointed at A, and the ends B and C are connected by a light inextensible string. The rods are kept in equilibrium in a vertical plane with the ends B and C on two smooth planes each of which inclined at an angle α to the horizontal; BC being horizontal and A being above BC. Find the reaction at B. If $\tan\theta > \tan 2\alpha$, where $B\hat{A}C = 2\theta$ then show that the tension in the string is $\frac{1}{2}W(\tan\theta - 2\tan\alpha)$. Find also the reaction at the joint A.



Let $2a$ be the length of each rod.

The system is symmetrical about the vertical axis through A.

Hence the vertical component of the reaction at A is zero.

For equilibrium of the system

Resolving vertically,

$$\uparrow\ 2R\cos\alpha - 2W = 0\ ;\ R = W\cos\alpha$$

For equilibrium of AB, Taking moment about A

$$\text{A} \curvearrowright \quad T.\,2a\cos\theta + R\sin\alpha\,.\,2a\cos\theta + W\,.\,a\sin\theta - R\cos\alpha\,.\,2a\sin\theta = 0$$

$$T = \frac{W}{2}(\tan\theta - 2\tan\alpha)$$

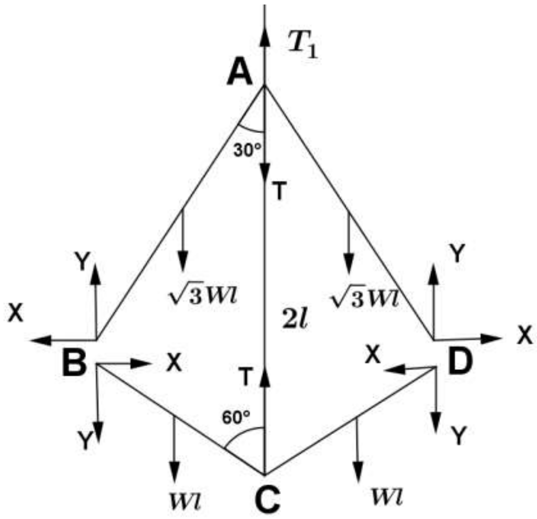
For equilibrium of AB,

$$\text{B} \curvearrowright \quad X\,.\,2a\cos\theta - W\,.\,a\sin\theta = 0$$

$$X = \frac{W}{2}\tan\theta$$

Example 10

AB, BC, CD and AD are four uniform rods having lengths $AB = AD = \sqrt{3}\ell$ and $BC = DC = \ell$ and are smoothly jointed at their ends to form a frame ABCD. The rods have weights W per unit length. The joints A and C are connected by an inelastic string of length 2ℓ. The frame is suspended in a vertical plane from A. Show that the tension in the string is $\frac{W\ell}{4}\left(\sqrt{3}+5\right)$



Method 1

$AB^2 + BC^2 = 3\ell^2 + \ell^2 = 4\ell^2 = AC^2$

Therefore, $A\hat{B}C = 90°$, $B\hat{A}C = 30°$, $B\hat{C}A = 60°$

The system is symmetrical about AC. Hence reactions at B and D are same.

For equilibrium of AB, Taking moment about A

$$\text{A}\curvearrowleft \quad X.\sqrt{3}\ell\cos 30° + Y.\sqrt{3}\ell\sin 30° - \sqrt{3}\ell W.\frac{\sqrt{3}}{2}\ell\sin 30° = 0$$

$$X.\cot 30° + Y = \frac{\sqrt{3}}{2}\ell W$$

$$\sqrt{3}X + Y = \frac{\sqrt{3}}{2}\ell W \quad \text{..............................} \ ①$$

For equilibrium of BC, moment about C

$$\text{C}\curvearrowleft \quad Y.\ell\sin 60° + W\ell.\frac{\ell}{2}\sin 60° - X.\ell\cos 60° = 0$$

$$Y + \frac{W\ell}{2} = \frac{X}{\sqrt{3}}$$

$$X = \sqrt{3}Y + \frac{\sqrt{3}W\ell}{2} \quad \text{....................} \ ②$$

Substitute ① and ②

$$Y + \sqrt{3}X = \frac{\sqrt{3}W\ell}{2}$$

$$Y + \sqrt{3}\left(\sqrt{3}Y + \frac{\sqrt{3}W\ell}{2}\right) = \frac{\sqrt{3}W\ell}{2}$$

$$4Y + \frac{3W\ell}{2} = \frac{\sqrt{3}W\ell}{2}$$

$$Y = \frac{W\ell}{8}\left(\sqrt{3}-3\right)$$

for equilibrium of BC and CD

$$\uparrow \ T - 2Y - 2W\ell = 0$$

$$T = 2Y + 2W\ell$$

$$T = 2\frac{W\ell}{8}\left(\sqrt{3}-3\right) + 2W\ell$$

$$T = \frac{W\ell}{4}\left(\sqrt{3}+5\right)$$

or For BC and CD take moments about D

Method 2

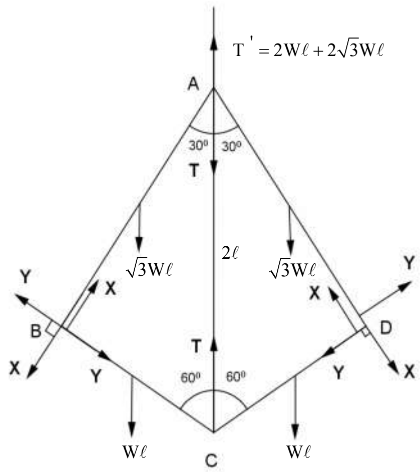
$AB^2 + BC^2 = 3\ell^2 + \ell^2 = 4\ell^2 = AC^2$

$A\hat{B}C = 90^0$, $B\hat{A}C = 30^0$, $B\hat{C}A = 60^0$

By symmetry reactions at B and D are same.

The components of the reaction at B are taken along BA and BC, since $A\hat{B}C = 90^0$

For the equilibrium of the rod AB,



$$\text{A}\circlearrowleft \quad \sqrt{3}W\ell.\ \frac{\sqrt{3}\ell}{2}\sin 30° - Y.\sqrt{3}\ell = 0$$

$$Y = \frac{\sqrt{3}W\ell}{4}$$

For the equilibrium of BC,

$$\text{C}\circlearrowleft \quad W\ell.\ \frac{\ell}{2}\sin 60 - X.\ell = 0$$

$$X = \frac{\sqrt{3}W\ell}{4}$$

For the equilibrium of BC and CD, Resolving vertically

$$T - 2W\ell + 2X\cos 30° - 2Y\cos 60° = 0$$

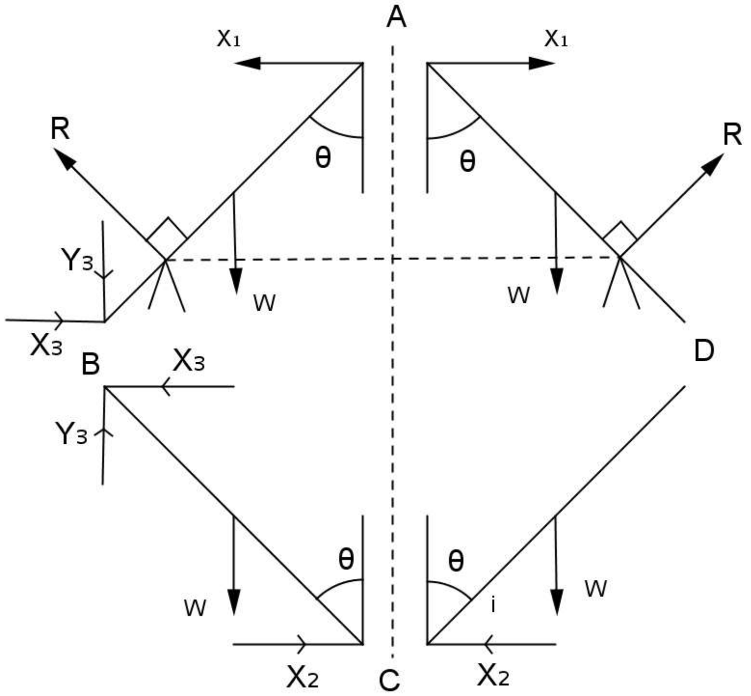
$$T = 2W\ell + 2Y\cos 60° - 2X\cos 30°$$

$$= 2W\ell + \frac{\sqrt{3}W\ell}{4} - \sqrt{3}.\frac{\sqrt{3}W\ell}{4}$$

$$= \frac{W\ell}{4}\left(\sqrt{3}+5\right)$$

Example 11

Four uniform rods AB, BC, CD, DA each of length $2a$ and weight W are freely hinged at their ends, and rest with the upper rods AB, AD in contact with two smooth pegs in the same horizontal line at a distance $2c$ apart. If the inclination of the rods to the vertical is θ, determaine the horizontal and vertical components of the reaction at B and prove that $c = 2a\sin^3\theta$.



The system is symmetrical about AC. Therefore, the vertical components of the reaction at A and C are zero.

For equilibrium of the system,

Resolving vertically,

$$\uparrow 2\text{R}\sin\theta - 4W = 0$$

$$\text{R} = \frac{2W}{\sin\theta} \text{ .... ①}$$

For equilibrium of BC,

Bm $\quad \text{X}_2.\,2a\cos\theta - W.a\sin\theta = 0$

$$\text{X}_2 = \frac{W\tan\theta}{2} \text{ .... ②}$$

Resolving horizontally,

$$\rightarrow \text{X}_2 - \text{X}_3 = 0;\ \ \text{X}_3 = \text{X}_2 = \frac{W\tan\theta}{2} \text{ ......③}$$

Resolving vertically

$$\uparrow \text{Y}_3 - \text{W} = 0\ ;\ \ \text{Y}_3 = \text{W} \text{ ....④}$$

For equilibrium of AB,

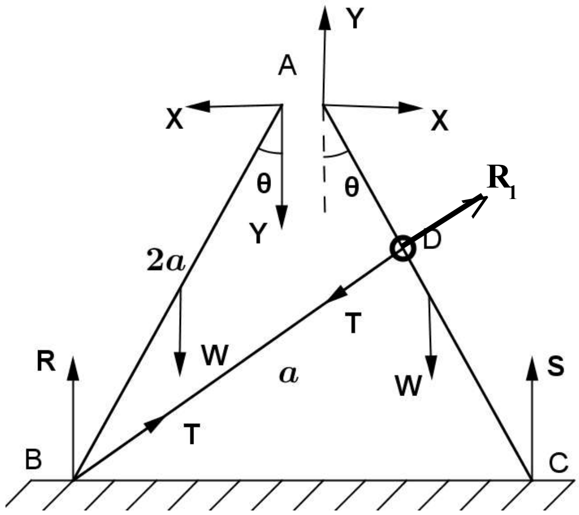
Am $\quad -\text{R}.\dfrac{\text{c}}{\sin\theta} + Wa.\sin\theta + \text{Y}_3.2a\sin\theta + \text{X}_3.2a\cos\theta = 0$

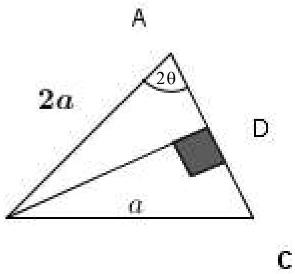
$$-\frac{2W.\text{c}}{\sin^2\theta} + \text{W}.a\sin\theta + W.2a\sin\theta + \frac{W}{2}.2a\sin\theta = 0\ ; \qquad c = 2a\sin^3\theta$$

Example 12

Two equal uniform rods AB, AC each of length $2a$ and weight W are smoothly jointed at A. BD is a weightless bar of length a, smoothly jointed at B and fastened at D to a small smooth light ring sliding on AC. The system is in equilibrium in a vertical plane with ends B and C resting on a horizontal plane. Show that the magnitude of the reaction at A is equal to $\frac{W}{12}\left(3\sqrt{2}-\sqrt{6}\right)$. Also show that the magnitude of the reaction at A is equal to the stress on BD and it makes an angle 15° with the horizontal.

Find the point where the line of action meets BC.





$A\hat{D}B = 90°$

$\sin 2\theta = \frac{1}{2}$

$2\theta = \frac{\pi}{6}$

$\theta = \frac{\pi}{12}$

For the equilibrium of the ring $R_1 = T$ and R_1 is perpendicular to AC, so T is perpenticular to AC

For the system

Resolve the forces vertically

$$\uparrow \quad R + S = 2W$$

By taking moment about C

$$Cm \quad W.a\cos 75° + W.3a\cos 75° = R.4a\cos 75° = 0$$

$$\Rightarrow R = W$$

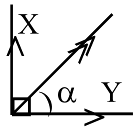
$$R = S = W$$

Consider the equilibrium of the rod AC

By taking moments about A for AC

$$Am \quad T.a\sqrt{3} + W.a\sin 15° - W.2a\sin 15° = 0$$

$$T = \frac{W\sin 15°}{\sqrt{3}} = \frac{W}{12}\left(3\sqrt{2}-\sqrt{6}\right)$$



For the rod AB resolve the forces horizontally and vertically

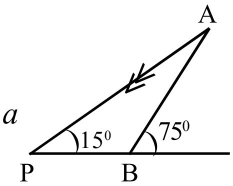
$$\rightarrow \quad X = T\cos 15° \;; \qquad \uparrow \quad Y = T\sin 15° \;;$$

$$A = \sqrt{X^2+Y^2} = T \;; \qquad \tan\alpha = \frac{X}{Y} = \tan 15°$$

$$\alpha = 15^0$$

Using sine rule in ΔABP $\quad \frac{BP}{\sin 60°} = \frac{AB}{\sin 45°} \Rightarrow BP = \frac{2a.\sin 60°}{\sin 15°}$

$$BP = \frac{2a.\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{2\sqrt{6}}{\sqrt{3}-1} \quad \Rightarrow BP = \left(3\sqrt{2}+\sqrt{6}\right)a$$



5.4 Exercises

1. Two uniform rods AB and AC of equal length are freely hinged at B. The weights of AB and BC are W_1 and W_2 respectively. The system freely hangs from fixed points B and C at the same level and BC = $2a$. If the depth of A below BC is a, find the horizontal and vertical components of the reaction at A.

2. Two equal uniform rods AB, BC each of weight W are smoothly jointed at B and their midpoints are connected by an inextensible string. The string has length such as when it is taut $A\hat{B}C$ makes 90°. The system is suspended from A freely while the string is taut. Show that the inclination of AB with the vertical at equilibrium is $\tan^{-1}\left(\frac{1}{3}\right)$ and the tension in the string is $\frac{3W}{\sqrt{5}}$. Also find the reaction on BC and show that it is in the direction of BC.

3. Two uniform equal rods AB, AC of length $2a$ and weight W, smoothly jointed at A lie symmetrically on the curved surface of a right circular cylinder whose axis is fixed horizontally. If each rod makes an angle θ with the horizontal and r is the radius of the cylinder, show that $r = a \operatorname{cosec}\theta \cos^3\theta$. Find also the reaction at A.

4. AB, BC and AC are three uniform equal rods smoothly jointed at ends A , B and C. AB and AC are each of weight W and the weight of BC is $2W$. The frame hangs freely from C. Show that BC makes an angle $\tan^{-1}\left(\frac{4}{\sqrt{3}}\right)$ with the horizontal. Find also the reaction at A and B.

5. Two uniform equal rods AOB and COD each of weight W are freely jointed at O, AO = CO = a, and BO = OD = $3a$. At equilibrium B and D rest on a horizontal plane and B, D are connected by an inextensible string of length $3a$. The system lies in equilibrium in a vertical plane. Show that the tension in the string is $\frac{2\sqrt{3}\,W}{9}$ and find the reaction at O.

6. Two uniform equal rods AB and AC of weight $2W$ and W, respectively, are smoothly jointed at A. B and C are fixed to a horizontal log. Find the horizontal and vertical components of the reaction at A. If the reaction at B and C are perpendicular to each other and $A\hat{B}C = \alpha$, show that $3\cot\alpha = \sqrt{35}$.

7. Three uniform equal rods OA, AB and BC each of length $2a$ and weight W are freely jointed at A and B. The end O is hinged to a fixed point and a horizontal force P is applied to BC at C and BC makes an angle 45° to the horizontal. Find P in terms of W. Show that the reaction at O is $\frac{\sqrt{37}\,W}{2}$. Show also that C is at a horizontal distance $2a\left[\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{10}}+\frac{1}{\sqrt{26}}\right]$ from the vertical through O.

8. Two equal uniform rods AB, BC each of length a and weight W are smoothly jointed at B. The rod AB is free to rotate about the point at which A is hinged. A small light ring is attached to C which is free to slide along another fixed rod through A. The fixed rod is inclined downwards, making an angle α to the horizontal. If the system is in equilibrium show that

(i) $\tan B\hat{A}C = \frac{1}{2}\cot\alpha$

(ii) The horizontal component of the reaction at B is $\frac{3W}{8}\sin 2\alpha$.

9. Four uniform equal rods AB, BC, CD and AD each of weight W are smoothly jointed at their ends to form a rhombus ABCD and hangs from A. The system is maintained in the shape of a square connecting the midpoints of BC and CD by a light rod. Find the thrust in the light rod and the reaction at C.

10. Five uniform equal rods AB, BC, CD, DE and EA each of weight W are freely jointed at their ends A, B, C, D and E to form a pentagon. The rods AB and AE make equal angles α and the rods BC and ED make equal angles β with the vertical. The system is hanged from A and the pentagon shape is maintained by connecting B and E by a light rod.

(i) Find the horizontal and vertical components of the reaction at C.

(ii) Show that the stress in BE is $W(\tan\alpha + \tan\beta)$.

(iii) Find the value of the stress when the pentagon is regular.

11. Four equal uniform rods AB, BC, CD and DA each of length $2a$ and weight W are smoothly jointed at A, B, C and D. The midpoints of BC and CD are connected by a light rod of length $2a\sin\theta$. The frame is freely hanged from A.

(i) Show that the thrust in the light rod is $4W\tan\theta$.

(ii) Find the reaction at B and C.

12. Four equal uniform rods AB, BC, CD and AD each of weight W are smoothly jointed at their end points to make a square ABCD. The frame is hanged from A. The shape is maintained by joining the midpoints of AB and BC by an inextensible string.

(i) Show that the reaction at D is horizontal and its magnitude is $\frac{W}{2}$

(ii) Show that the tension in the string is $4W$

(iii) Show that the reaction at C is $\frac{W\sqrt{5}}{2}$ and it makes an angle $\tan^{-1}\left(\frac{1}{2}\right)$ with the vertical.

(iv) Show that the reaction at B is $\frac{W\sqrt{17}}{2}$ and it makes an angle $\tan^{-1}\left(\frac{1}{4}\right)$

13. Four uniform equal rods AB, BC, CD and DA each of weight W are smoothly jointed at the ends to form a square ABCD. The frame is suspended from A and a weight $3W$ is attached to the point C. The shape is maintained by connecting the midpoints of AB and AD with a light rod. Show that the thrust in the light rod is $10W$.

14. Four uniform rods of equal length ℓ and weight W are freely jointed to form a framework ABCD. The joints A and C are connected by a light elastic string of natural length a. The framework is freely suspended from A and takes the shape of a square. Find the modulus of elasticity of the string. Find also the reaction at the joints B and D.

15. Six uniform equal rods each of weight W are smoothly jointed at their end points to form a hexagon ABCDEF. The system is suspended from A and the shape is maintained by light rods BF and CE. Show that the stress in BF is five times the stress in CE.

16. A uniform rod is cut into three parts AB, BC and CD of lengths ℓ, 2ℓ and ℓ respectively. They are smoothly jointed at B and C and rest on a fixed smooth sphere whose radius is 2ℓ and centre O, so that the middle point of BC and the extremities A and D are in contact with the sphere. Show that the reaction on the rod BC at its mid point is $\frac{91W}{100}$ where W is the weight of the rod.
Find the magnitude and the direction on the rod CD at the joint C and the point whose line of action meets OD.

17. Three uniform rods AB, BC and AC of equal length a and weight W are freely jointed together to form a triangle ABC. The framework rests in a vertical plane on smooth supports at A and C so that AC is horizontal and B is above AC. A mass of weight W is attached to a point D on AB where $\mathrm{AD} = \frac{a}{3}$. Find the reaction at joint B.

18. Two uniform equal rods AB and AC each of weight W and length $2a$ are freely jointed at A and placed in a vertical plane with ends B and C on a smooth horizontal table. Equilibrium is maintained by a light inextensible string which connects C to the mid point of AB with each rod making an angle $\alpha\left(<\frac{\pi}{2}\right)$ with the horizontal. Show that the tension T in the string is $\mathrm{T} = \frac{W}{4}\sqrt{1+9\cot^2\alpha}$. Find the magnitude and the direction of the reaction at A.

19. Five uniform equal rods each of weight W are smoothly jointed at their ends to form a regular pentagon. CD is placed on a horizontal plane so that the frame is in a vertical plane and the shape is maintained by joining the midpoints of BC and DE by a light rod. Find the reaction at B and show that the tension in the light rod is $\left[\cot\frac{\pi}{5}+3\cot\frac{2\pi}{5}\right]W$.

20. Three equal uniform rods AB, BC, CD each of length $2a$ and weight W are smoothly jointed at B and C, and rest with AB, CD in contact with two smooth pegs at the same level. In the position of equilibrium AB and CD are inclined at an angle α to the vertical BC being horizontal. Prove that the distance between the pegs is $2a\left(1+\frac{2}{3}\sin^3\alpha\right)$. If β is the angle which the reaction at B makes with the vertical, prove that $\tan\alpha \, . \tan\beta = 3$.

6.0 Framework

In this chapter we will consider a framework consists of light rods joined at their ends to other rods with smooth joints.

6.1 Rigid Frame

If the shape of a frame is unaltered by external forces, then the frame in called a rigid frame.

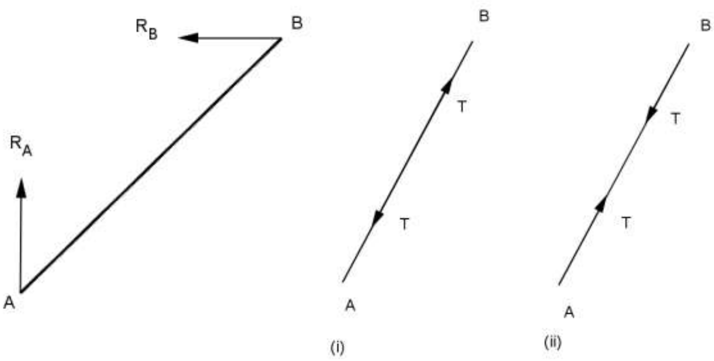
In a frame made by light rods, the reactions at the joints will act along the rods. These reactions along the rods are known as stresses.

If we consider a light rod AB in a frame R_A and R_B are the reactions at the joints by pins. The rod is in equilibrium under the action of these two forces R_A and R_B. Hence for the equilibrium of the rods R_A and R_B must be equal and act opposite along the rod.

$R_A = R_B = T$

(i) T is tension

(ii) T is thrust



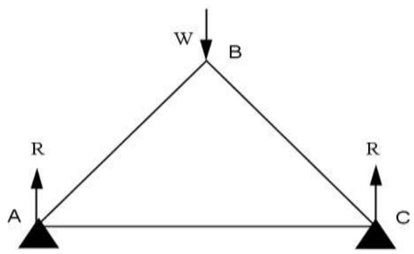
Assumptions when solving framework problems

- All the rods in the framework are light rods.
- All the rods are freely (smoothly) jointed at their ends and no couple in formed at a joint.
- The reactions at the joints (except external forces) will act along the rods. These may be thrusts or tensions.
- All the rods in the frame are in the same vertical plane and all the forces (including the external forces) are coplanar forces.
- External forces are applied only on joints.

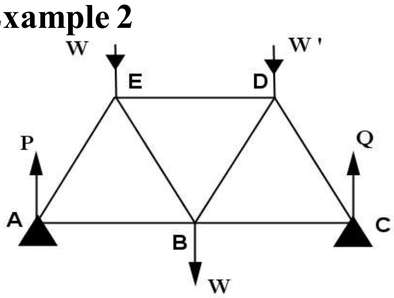
6.2 Representing external forces in a light framework in equilibrium

Example 1

Δ ABC is frame lying on A and C, carries a load W at B. By symmetry the reactions at A and C are equal.



Example 2



ABCDE is a frame made of seven equal light rods and rests on two pegs at A and C. It carries W at E, B and W′ at D. The external forces P, Q will be vertical.

Bow's Notation

- This notation is introduced by a mathematician called Bow.
- All the external forces will be represented outside of the frame.
- The region between forces (open or closed) is denoted by a small letter of the English alphabet or a number.
- Each force denoted by two letters of the alphabet belongs to the two region formed by the force.

Solving problems using Bow's notation

(i) Having represented all external forces and regions, forces of polygons have to be drawn for each joint of the frame (These polygon of forces will be a closed figure, the vertices of the polygon being denoted by the names of the letters of the regions.)

(ii) The values of the stresses in rods can be calculated by using trigonometric ratios and algebraic equations in the triangles and polygons obtained in the stress diagram.

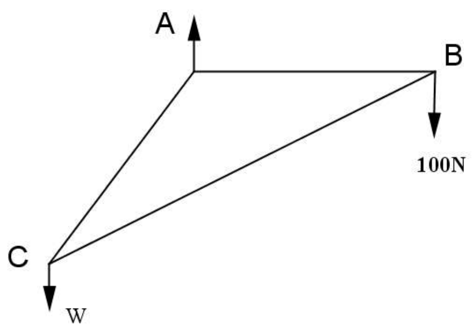
(iii) By reading the names of the sides in the stress diagram, mark the directions of the stress by using arrow marks.

(iv) While drawing force polygons, the disense has to be same for all the joints. (either clockwise or anticlockwise)

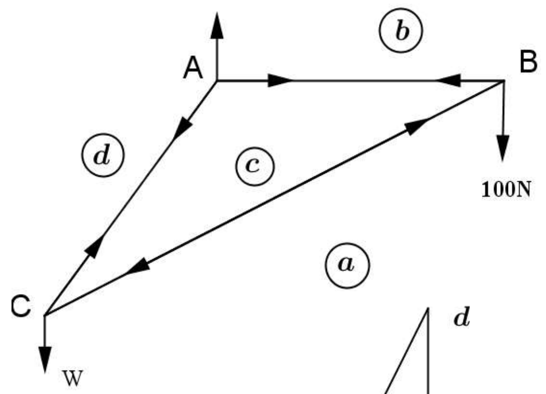
(v) To draw a polygon of forces at a joint there may be maximum of two unknown forces.

6.3 Worked examples

Example 1

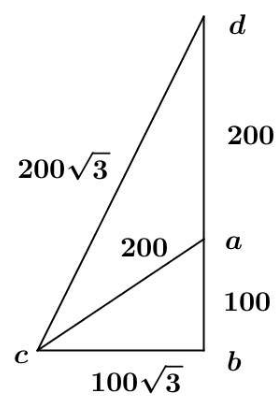


In the given figure, ABC is a triangular framework consisting of three smoothly jointed light rods AB, BC, CA, where AB = AC and $B\hat{A}C = 120°$. The framework is in a vertical plane with AB horizontal. It is supported at A by a smooth peg and carries loads 100 N at B and W N at C. Draw a stress diagram using Bow's notation and from it, calculate the stresses in the rods, distinguishing between tensions and thrusts and also find the value of W.



Start from joint B

Joint	Order	Name of Polygon
B	$a \to b \to c \to a$	Δabc
C	$a \to c \to d \to a$	Δbcd



AB(bc) = Tension $= 100\sqrt{3}$ N

BC(ca) = Thrust = $200\sqrt{3}$ N

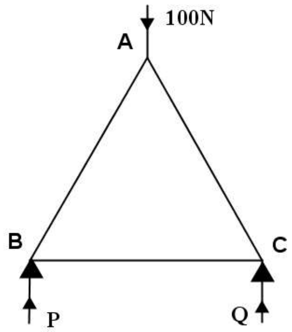
CA(cd) = Tension = $200\sqrt{3}$ N

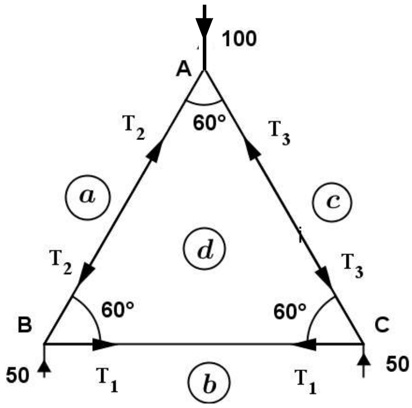
W(ad) = 200 N

In this problem all the joints are taken in the anticlockwise sense.

Example 2

ABC is a frame obtained by joining three uniform equal light rods AB, BC and AC. B and C rest on 2 pegs at the same horizontal level. A carries a load of 100N. Find the reaction at B and C. Draw a stress diagram by using Bow's notation. Hence find the stress in each rod distinguishing between tension and thrust.





For equilibrium

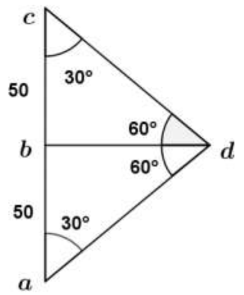
Resolve the forces vertically

$$\uparrow P + Q = 100$$

$$P = Q = 50 \quad \text{(symmetry)}$$

Polygon of forces has to be drawn for joints A, B and C by naming the regions between the vertices as a,b,c and d.

Stress diagram



This diagram is drawn by taking the region around each joint in anticlockwise disense starting from C.

Joint C $\rightarrow$ Joint A $\rightarrow$ Joint B

Joint	order	Name of Polygon
C	$b \rightarrow c \rightarrow d \rightarrow a$	Δbcd
A	$d \rightarrow c \rightarrow a \rightarrow d$	Δacd

Tensions and thrusts are denoted by naming the regions.

$$T_1 = bd = 50 \tan 30° = \frac{50}{\sqrt{3}} \text{ N}$$

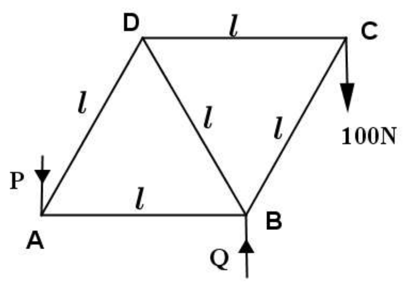
$$T_3 = cd = 50 \sec 30° = \frac{100}{\sqrt{3}} \text{ N}$$

$$T_2 = ad = 50 \sec 30° = \frac{100}{\sqrt{3}} \text{ N}$$

Rod	Stress	Thrust	Tension
AB	$\frac{100}{\sqrt{3}}$ N	✓	-
BC	$\frac{50}{\sqrt{3}}$ N	-	✓
AC	$\frac{100}{\sqrt{3}}$ N	✓	-

Example 3

The given figure represents the framework of five equal light rods. This frame is supported by a peg at B and a vertical force P is applied at A. C carries a load of 100 N. Find the stresses in each rod by drawing a stress diagram.



For equilibrium

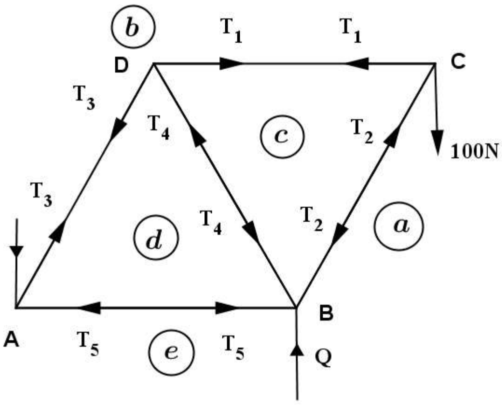
Resolve the forces vertically

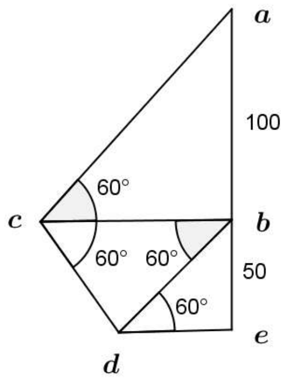
$\uparrow \; 100 + P = Q$ .... ①

By taking moments about A

A↻ $Q.2\ell = 100\,(2\ell + 2\ell \cos 60°)$ ....②

① and ② $\Rightarrow$ P = 50 N , Q = 150 N





In the above diagram regions are named starting from C and the stress diagram is drawn as follows.

Joint	order	Name of Polygon
C	$a \to b \to c \to a$	Δabc
D	$c \to d \to c \to b$	Δbcd
A	$d \to b \to e \to d$	Δdbe
B	$c \to d \to e \to a \to c$	$\square acde$

The force polygon is drawn starting from joint C joining the region in the anticlockwise disense.

$T_1 = bc = 100 \tan 30° = \dfrac{100\sqrt{3}}{3}$ N

$T_2 = ac = 100 \sec 30° = \dfrac{200\sqrt{3}}{3}$ N

$T_3 = bd = 50 \operatorname{cosec} 60° = \dfrac{100\sqrt{3}}{3}$ N

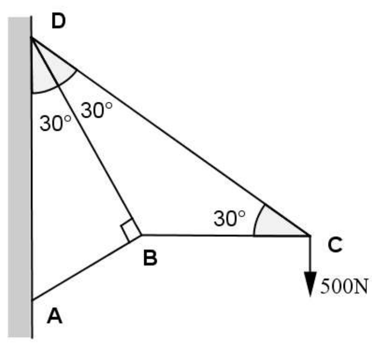
$T_4 = cd = bd = \dfrac{100\sqrt{3}}{3}$ N

$T_5 = dc = 50 \tan 30° = \dfrac{50\sqrt{3}}{3}$ N

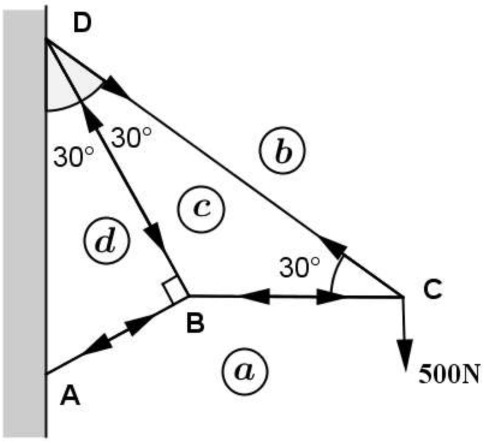
Rod	Stress	
DC	$\dfrac{100\sqrt{3}}{3}$ N	Tension
BC	$\dfrac{200\sqrt{3}}{3}$ N	Thrust
AD	$\dfrac{100\sqrt{3}}{3}$ N	Tension
BD	$\dfrac{100\sqrt{3}}{3}$ N	Thrust
AB	$\dfrac{50\sqrt{3}}{3}$ N	Thrust

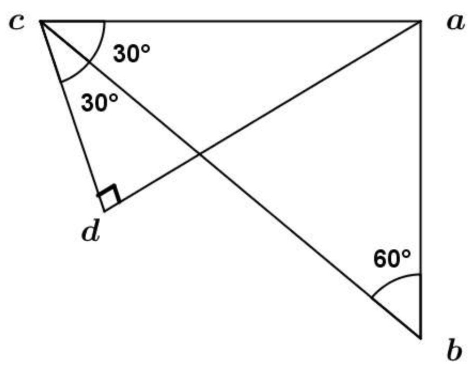
Example 4

A framework formed by four light rods AB, BC, CD and BD is shown in the given diagram. A, D are freely jointed to a vertical wall. Joint C carries a load of 500 N and BC remains horizontal. Draw a stress diagram using Bow's notation and find the stresses in each rod distinguishing between tensions and thrusts.



At joint C one force is known and two forces unknown. Draw the stress diagram starting from joint C.





Joint	order	Name of Polygon
C	$a \rightarrow b \rightarrow c \rightarrow a$	Δabc
B	$a \rightarrow c \rightarrow d \rightarrow a$	Δacd

bc = 500 sec 60° = 1 000 N

ac = 500 tan 60° = $500\sqrt{3}$ N

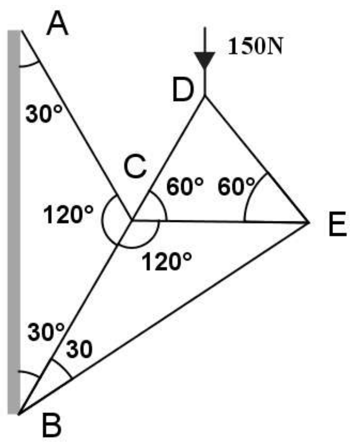
cd = ($500\sqrt{3}$ N) sin 30° = $250\sqrt{3}$ N

ad = $500\sqrt{3}$ N cos 30° = 750 N

Rod	Stress	Thrust	Tension
DC	1 000 N	-	✓
BC	$500\sqrt{3}$ N	✓	-
BD	$250\sqrt{3}$ N	✓	-
AB	750 N	✓	-

Example 5

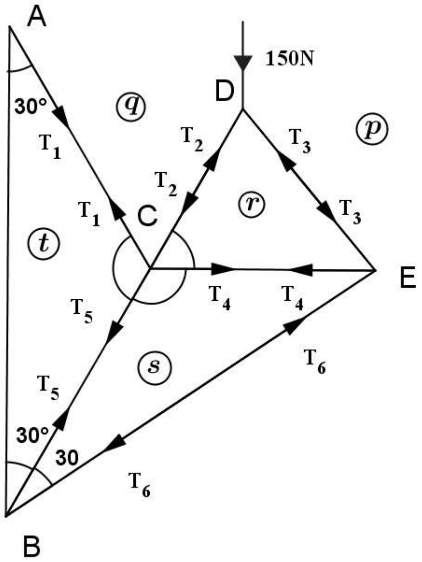
The given figure show a framework of six light rods smoothly jointed at C, D and E. A and B are smoothly jointed to a vertical wall and D carries a load of 150N. Draw a stress diagram using Bow's notation and find the stresses in each rod distinguishing between thrusts and tensions.

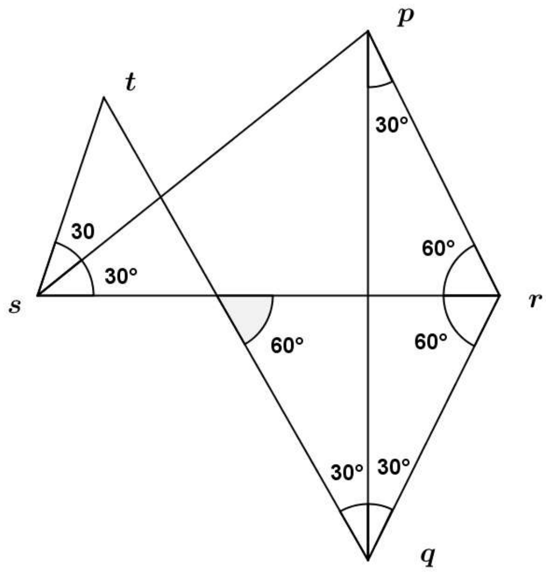


D is the joint with one known and two unknown forces

So start to draw triangle of forces from joint D

Joint	order	Name of Polygon
D	$p \rightarrow q \rightarrow r \rightarrow p$	Δpqr
E	$p \rightarrow r \rightarrow s \rightarrow p$	Δprs
C	$r \rightarrow q \rightarrow t \rightarrow s \rightarrow r$	$\square rqts$





$AC = qt = 75\sqrt{3} + 25\sqrt{3} = 100\sqrt{3}$ N

$CD = qr = 75 \sec 30° = 50\sqrt{3}$ N

$DE = pr = qr = 50\sqrt{3}$ N

$CE = sr = 100\sqrt{3}$ N

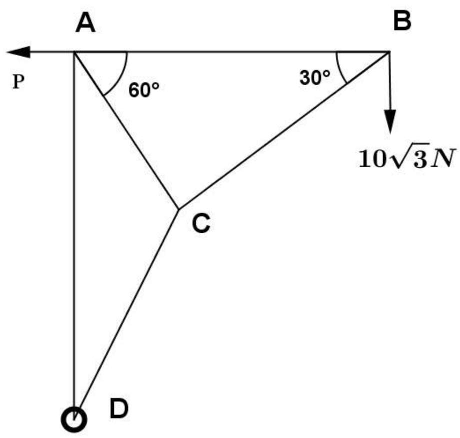
$BC = st = 50\sqrt{3}$ N

$BE = ps = 150\sqrt{3}$ N

Rod	Stress	Thrust	Tension
AC	$100\sqrt{3}$ N	-	✓
CD	$50\sqrt{3}$ N	✓	-
DE	$50\sqrt{3}$ N	✓	-
CE	$100\sqrt{3}$ N	-	✓
BC	$50\sqrt{3}$ N	-	✓
CE	$150\sqrt{3}$ N	-	✓

Example 6

Five rods AB, BC, CD, DA and AC are smoothly jointed at their ends to form a framework as shown in the figure. $A\hat{B}C = A\hat{D}C = D\hat{A}C = 30°$ and $B\hat{A}C = 60°$. The framework is smoothly hinged at D and carries a weight $10\sqrt{3}$ N at B. The framework is held in a vertical plane with AB horizontal by a horizontal force P at A.



(i) Find the value of P

(ii) Find the magnitude and direction of the reaction at D.

(iii) Using Bow's notation, draw a stress diagram for the framework and find the stresses in all the rods, distinguishing between tensions and thrusts.

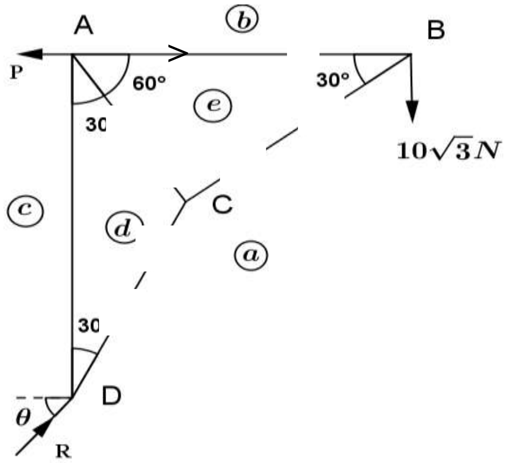
(i) For equilibrium

Take moment about D

Dm $\quad P.AD - 10\sqrt{3}\,AB = 0$

$$\text{but } AD = 2\,AC \cos 30°$$
$$= 2\,AB \cos 60° \cos 30°$$
$$AD = \frac{\sqrt{3}}{2} AB$$
$$\therefore \quad P.\frac{\sqrt{3}}{2} AB = 10\sqrt{3}\,AB$$
$$\therefore \quad P = 20 \text{ N}$$

Let R be the reaction at D and θ be the angle that R makes with the horizontal



Resolving vertically

$$\uparrow R\sin\theta = 10\sqrt{3}$$

Resolving horizontally

$$\rightarrow R\cos\theta = P = 20 \text{ N}$$

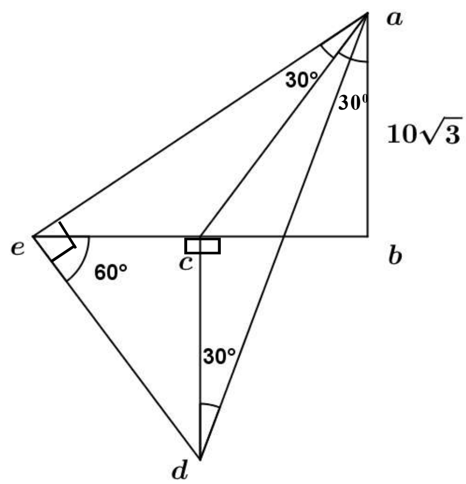
$$R = \sqrt{\left(10\sqrt{3}\right)^2 + 20^2} = 10\sqrt{7}$$

$$\tan\theta = \frac{10\sqrt{3}}{20} = \frac{\sqrt{3}}{2} \; ; \qquad \theta = \tan^{-1}\frac{\sqrt{3}}{2}$$

Since the system is in equilibrium under three forces the reaction R should also pass through B.

Start from joint B in the anticlockwise direction.

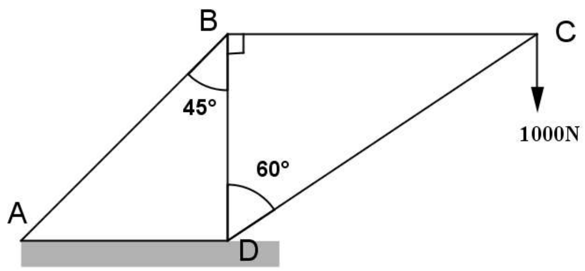
Joint	order	Name of Polygon
B	$a \rightarrow b \rightarrow e \rightarrow a$	Δabd
C	$a \rightarrow e \rightarrow d \rightarrow a$	Δaed



Rod	Stress	Magnitude
AB	Tension	30 N
BC	Thrust	$20\sqrt{3}$ N
AC	Thrust	20 N
DC	Thrust	40 N
AD	Tension	$10\sqrt{3}$ N

$$be = 10\sqrt{3}\tan 60 = 30$$
$$ae = 10\sqrt{3}\sec 60 = 20\sqrt{3}$$
$$ad = 20\sqrt{3}\sec 30 = 40$$
$$ed = ae\tan 30 = 20$$
$$cd = ed\sin 60 = 10\sqrt{3}$$

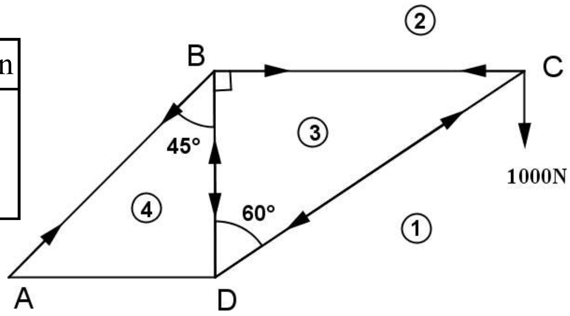
Example 7



The given figure shows a crane composed of four freely jointed rods AB, BC, CD and BD. The rod BC is horizontal while the rod BD is vertical. The crane is fixed to the horizontal ground at A and D and there is a load of 1 000 N hanging at C. Use Bow's notation to find the forces in the rods, distinguishing between tensions and thrusts.

Start with joint C in anticlockwise

Joint	order	Name of Polygon
C	$1 \to 2 \to 3 \to 1$	$\Delta 123$
B	$3 \to 2 \to 4 \to 3$	$\Delta 324$



$AB = km = 1000\sqrt{6}$ N

$BC = kl = 1\,000 \cot 30^0 = 1\,000\sqrt{3}$ N

$CD = lj = 1\,000 \operatorname{cosec} 30° = 2\,000$ N

$BD = ml = kl = P.\ell \cos 30° - 10\sqrt{3}\,.\,2\ell = 0$

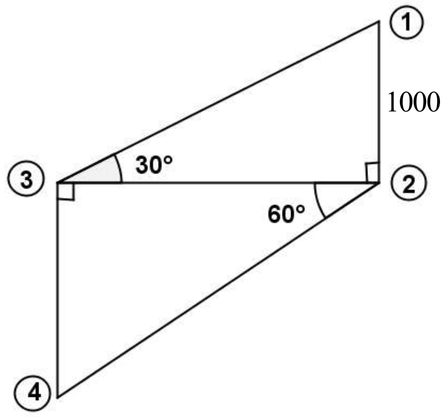
$\Rightarrow P = 40$ N

$\rightarrow\ P = R \cos\theta = 40$ N

$\uparrow R \sin\theta = 10\sqrt{3}$ N

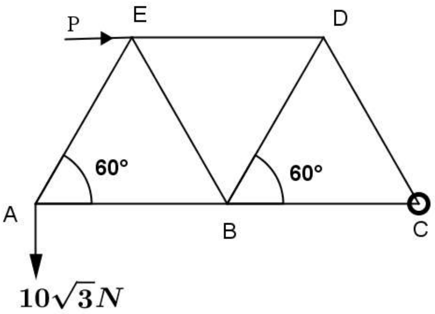
$R = \sqrt{40^2 + (10\sqrt{3})^2}$ N

$R = 10\sqrt{19}$ N



Rod	Stress	Thrust	Tension
AB	$1000\sqrt{6}$ N	-	✓
BC	$1000\sqrt{3}$ N	-	✓
CD	2 000 N	✓	-
BD	$1000\sqrt{3}$ N	✓	-

Example 8



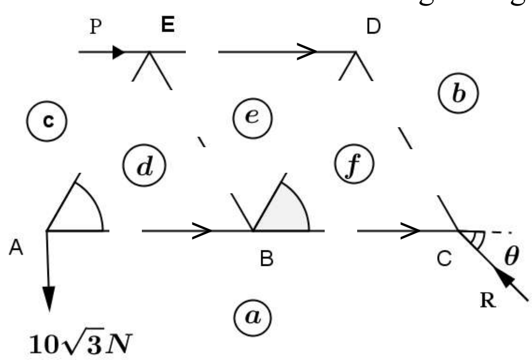
The given figure shows a framework consisting of seven light rods AB, BC, CD, DE, EA, EB and BD smoothly jointed at their extremities. The frame smoothly jointed at C and carries a load $10\sqrt{3}$ N at A. A horizontal force P at E keeps ED horizontal and the frame is in a vertical plane.

(i) Find the value of P at E

(ii) Find the magnitude and the direction of the reaction at C

(iii) Draw a stress diagram using Bow's notation and hence find the stresses in each rod distinguishing between tensions and thrusts.

(iv) From stres diagram verify reaction at C



For equilibrium

Take moment about C

n $P.\ell \cos 30^0 - 10\sqrt{3}.2\ell = 0$,where ℓ is length of a rod.

$\Rightarrow P = 40N$

Resolve the forces in the horizontal direction

$$\rightarrow \; P = R\cos\theta = 40N$$

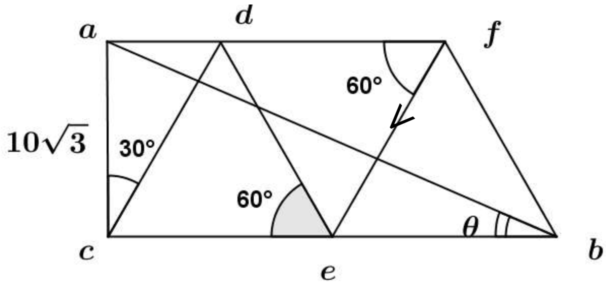
Resolve vertically

$$\uparrow R\sin\theta = 10\sqrt{3}N$$

$$R = \sqrt{40^2 + \left(10\sqrt{3}\right)^2}$$

$$R = 10\sqrt{19}N$$

$$\tan\theta = \frac{10\sqrt{3}}{40} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{\sqrt{3}}{4}\right)$$



Stress diagram

Start from joint A clockwise

Joint	order	Name of Polygon
A	$c \rightarrow a \rightarrow d \rightarrow c$	Δcad
E	$c \rightarrow d \rightarrow e \rightarrow b \rightarrow c$	$\square cdeb$
D	$b \rightarrow e \rightarrow f \rightarrow b$	Δbef

Join af, bf

$AB = ad = 10\sqrt{3}\tan 30° = 10$ N

$AE = cd = 10\sqrt{3}\sec 30° = 20$ N

$cd = de = 20$ N

$AE = BE = 20$ N

$bf = de = ef = df = 20$ N

$CD = DE = BD = 20$ N

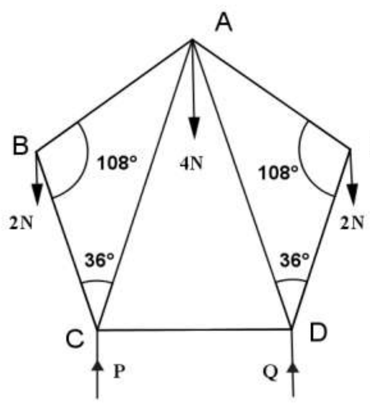
Reaction at c denoted by ab

$ab^2 = \left(10\sqrt{3}\right)^2 + 40^2$

$ab = 10\sqrt{19}$

Rod	Stress	
AB	10 N	Thrust
BC	30 N	Thrust
CD	20 N	Thrust
DE	20 N	Thrust
EA	20 N	Tension
EB	20 N	Thrust
DB	20 N	Tension

Example 9



A framework of seven freely jointed light rods is in the form of a regular pentagon ABCDE and the diagonals AC and BD. The framework is in the vertical plane with the lowest rod CD horizontal and is supported at C and D by two upward vertical forces of magnitude P and Q and weights 2 N, 4 N, 2 N are suspended at B, A and E respectively. Draw a stress diagram for this framework using Bow's notation. Hence determine the stresses in all seven rods, distinguishing between tensions and thrusts. Give the answers in terms of $\cos\frac{n\pi}{10}$ where n is a positive integer.

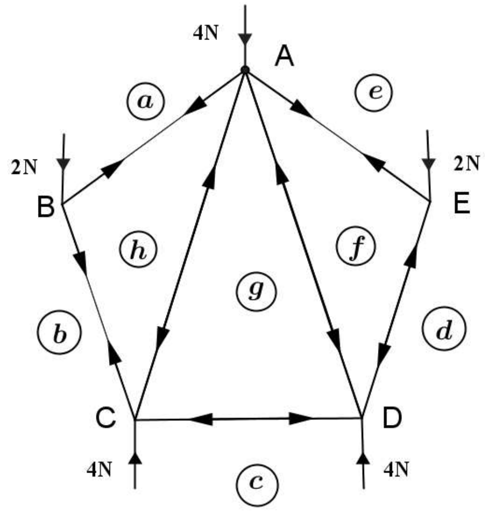
For the equilibrium of the system

Resolve the forces vertically

$$P + Q = 8\text{ N}$$

$$\uparrow \quad P = a \qquad \text{(symmetry)}$$

$$P = Q = 4\text{ N}$$



The system is symmetrical about the vertical line through A.

Start from the joint B and move in the clockwise direction.

$$n = 18° \;(\text{ say }) = \frac{\pi}{10}$$

de = ec = ca = ab = 2 N

let gc = x

Then pc = $x \tan 4n°$

AB(bc) = Tension $= 100\sqrt{3}$ N

BC(ca) = Thrust $= 200\sqrt{3}$ N

CA(cd) = Tension $= 200\sqrt{3}$ N

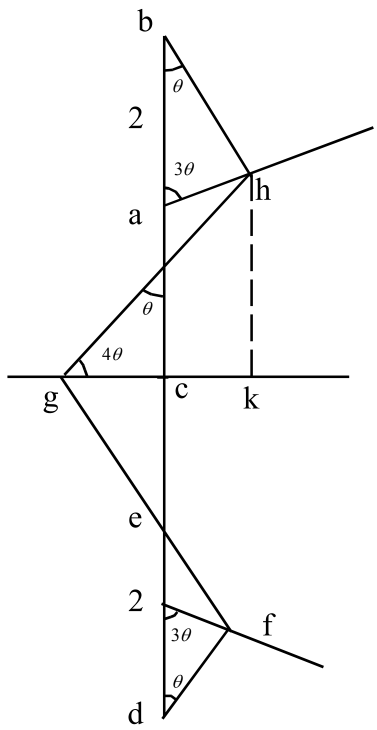
W(ad) = 200 N

First draw a vertical line and denote the vertical forces, in clockwise sense as

- ba, ae, ed, dc, ca

$$\theta = \frac{\pi}{10} = 18^0$$

Joint	Order	Polygon
B	$b \to a \to h \to b$	Δ bah
E	$e \to d \to f \to e$	Δ edf
A	$h \to a \to e \to f \to g \to h$	⬠ haefg
C	$b \to h \to a \to c \to h$	□ bhgc



In Δabh, Using sine rule

$$\frac{ah}{\sin\theta} = \frac{2}{\sin 4\theta} = \frac{bh}{\sin 3\theta}$$

$$ah = 2\frac{\sin\theta}{\sin 4\theta}\ ,$$

$$bh = 2\frac{\sin 3\theta}{\sin 4\theta}$$

Δ 's abh and def are concruent

$$\therefore af = ah = \frac{2\sin\theta}{\sin 4\theta}$$

$$\therefore df = bh = \frac{2\sin 3\theta}{\sin 4\theta}$$

In Δ ghk

$$hk = gh\sin 4\theta = ac + ah\cos 3\theta$$

$$gh = \frac{2}{\sin 4\theta} + 2\frac{\sin\theta}{\sin 4\theta}.\cos 3\theta$$

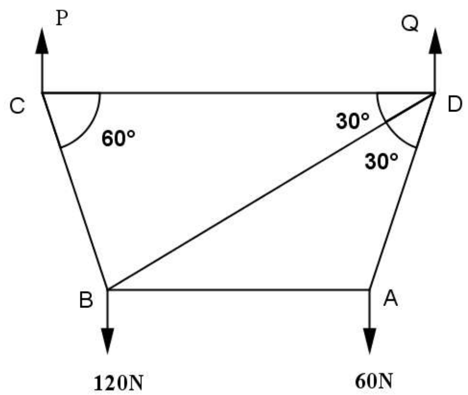
$$gc + ah\sin 3\theta = gh\cos 4\theta$$

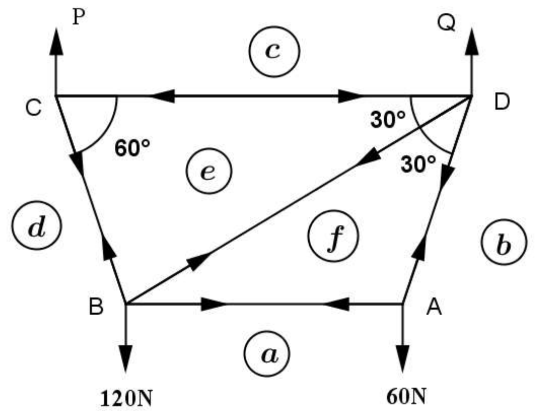
$$gc = gh\cos 4\theta - ah\sin 3\theta$$

Rod	stress	Tension	Thrust
AB	ah	✓	-
BC	bh	-	✓
AE	ah	✓	-
ED	bh	-	✓
AC	gh	-	✓
AD	gh	-	✓
DC	gc	✓	-

Example 10

The given frame consists of five light rods AB, AD, BC, BD and CD smoothly jointed at their extremities while the frame carries 120 N and 60 N at B and A respectively, the vertical forces P Newton and Q Newton applied at C and D to make AB and CD horizontal. Draw a stress diagram using Bow's notation and hence find the stresses in all five rods distinguishing between thrusts and tensions.



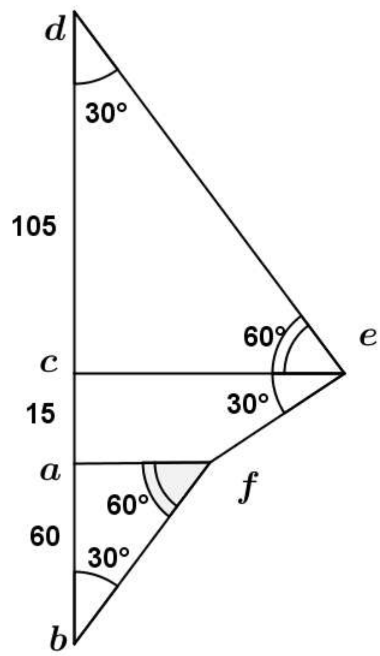


For equilibrium

Resolving vertically

$$\uparrow P + Q - 120 - 60 = 0$$

$$P + Q = 180N$$



Taking moments about D

$$P.\ 2\ell - 60.\ell\cos 60 - 120.\ell + \ell\cos 60 = 0$$

$$2P = 30 + 180$$

$$\Rightarrow P = 105N$$

$$\Rightarrow Q = 180 - 105 = 75N$$

Stress diagram

Start from joint C and move anticlockwise

C → B → A

Step I : Cm   Step II : Bm   Step III : Am

$AB = af = 60 \tan 30° = 20\sqrt{3}$ N

$BC = ed = 105 \sec 30° = 70\sqrt{3}$ N

$CD = ec = 105 \tan 30° = 35\sqrt{3}$ N

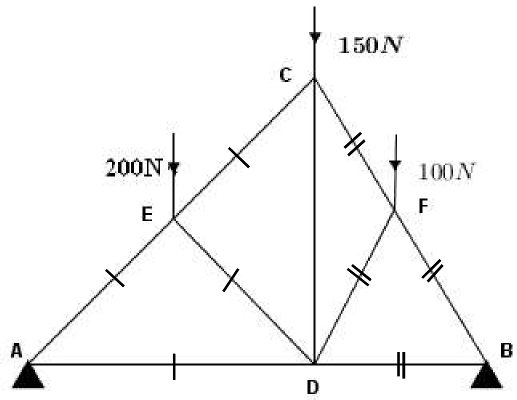
$AD = bf = 60 \sec 30° = 40\sqrt{3}$ N

$BD = ef = 15 \sec 30° = 10\sqrt{3}$ N

Rod	Stress	
AB	$20\sqrt{3}$ N	Tension
BC	$70\sqrt{3}$ N	Tension
CD	$35\sqrt{3}$ N	Thrust
AD	$40\sqrt{3}$ N	Tension
BD	$10\sqrt{3}$ N	Tension

6.4 Exercises

1.

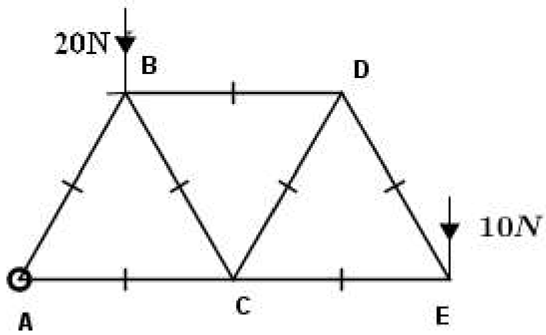


The figure represents the framework of a roof whose weight may be regarded as distributed in the manner shown above.

i. Find the reaction at A and B.

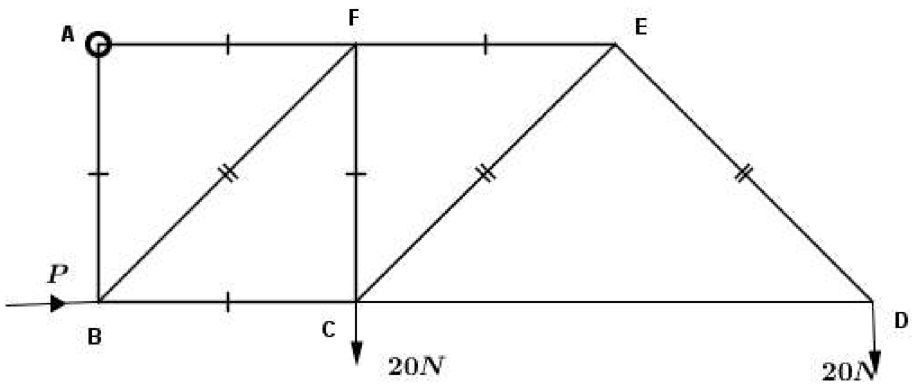
ii. Draw the stress diagram by using Bow's notation and find the stress in each rod, distinguishing between tensions and thrusts.

2.



The above figure shows a framework made by seven light rods. The frame is hinged at A to a fixed point and kept in position by a horizontal force P at B. Draw a stress diagram using Bow's notation and find the stress in each rod distinguishing between tensions and thrusts.

3.

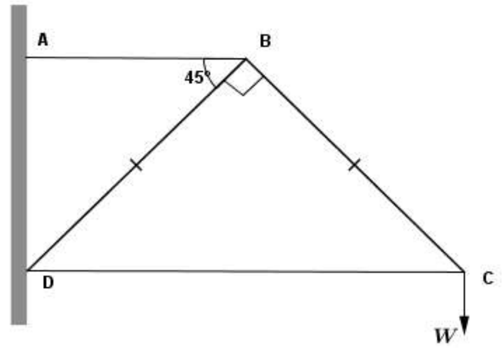


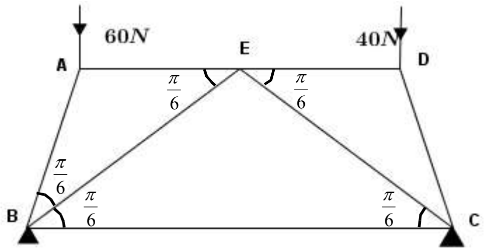
The above framework consists of nine smoothly jointed light rods, smoothly hinged to a fixed point at A, kept in equilibrium by a horizontal force P at B and loaded with 20 W each at C and D.

i. Find P and the reaction at A.

ii. Draw the stress diagram using Bow's notation and find the stress in each rod, distinguishing between tensions and compressions.

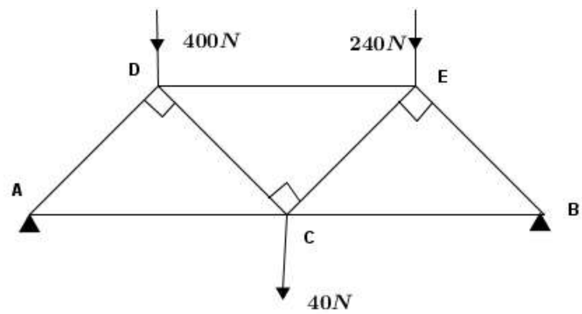
4. The framework consists of four light rods AB, BC, CD and DB freely jointed at B, C, D and attached to a vertical wall at A, and D loaded with WN at C. Draw a stress diagram using Bow's notation and find the stress in each rod, distinguishing between tensions and thrusts.

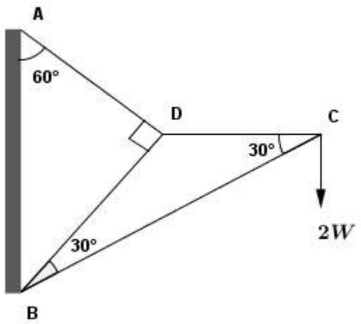


5. 

ABCDEF is a framework which has freely jointed rods AB, BC, CD, DE, AE, BE and CE such that $E\hat{B}C = E\hat{C}B = A\hat{B}E = D\hat{C}E = A\hat{E}B = D\hat{E}C = \frac{\pi}{6}$. The framework is supported at B and C such that BC is horizontal, and loaded 60 N, 40 N at A and D respectively. Draw a stress diagram using Bow's notation and find the stress in each rod, distinguishing thrusts and tensions.

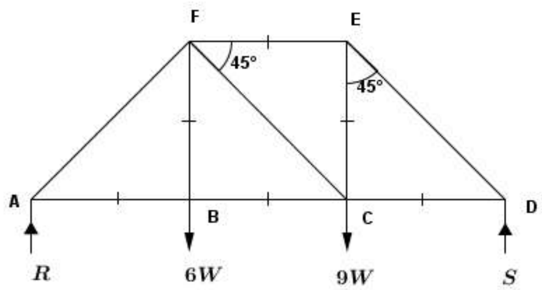
6. The framework in the figure is formed by using light bars according to the diagram. All triangles are right angular and isosceles. The system is on support at A and B such that ACB horizontal. The framework carries loads of 40 N, 400 N and 240 N at C, D and E respectively. Draw stress diagram using Bow's notation and find the stresses in the bars distinguishing tensions and thrusts.



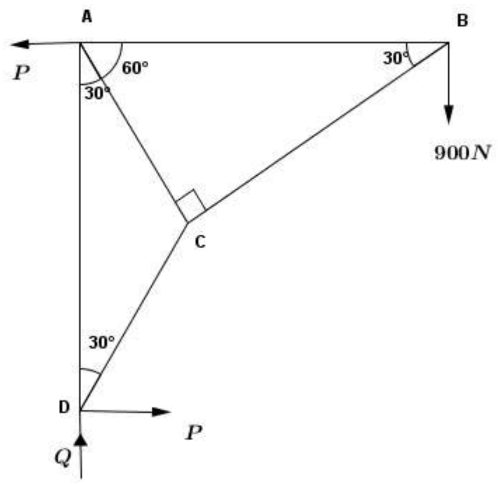
7. 

The figure shows a framework consisting freely jointed four light bars AD, BD, BC and CD. It is hinged freely to a vertical wall at A and B. C carries a load of $2W$. By using stress diagram find the reactions at A and B. Hence find the stresses in the rods distinguishing tensions and thrusts.

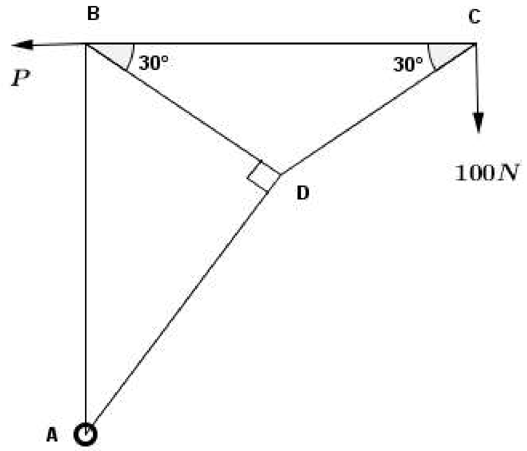
8. The figure shows a framework consisting nine light rods freely jointed at A, B, C, D, E and F. The frame carries loads of $6W$ and $9W$ at B and C respectively. It is supported by vertical forces R and S at A and D respectively. Draw a stress diagram and find the stresses in the rods distinguishing between tensions and thrusts.



9. A freely jointed framework consisting of five light rods is shown in the figure. Joint B carries a load of 900N. The framework is in equilibrium such as AD is vertical by means of forces P and (P, Q) acting on A and D respectively (P is horizontal and Q is vertical). Find the magnitudes of forces P and Q. Draw the stress diagram using Bow's notation and find the stress in each rod, distinguishing tensions and thrusts.



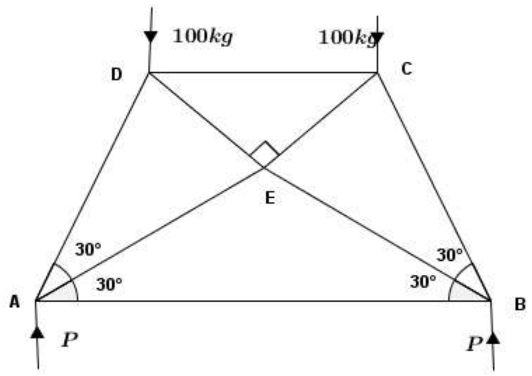
10.



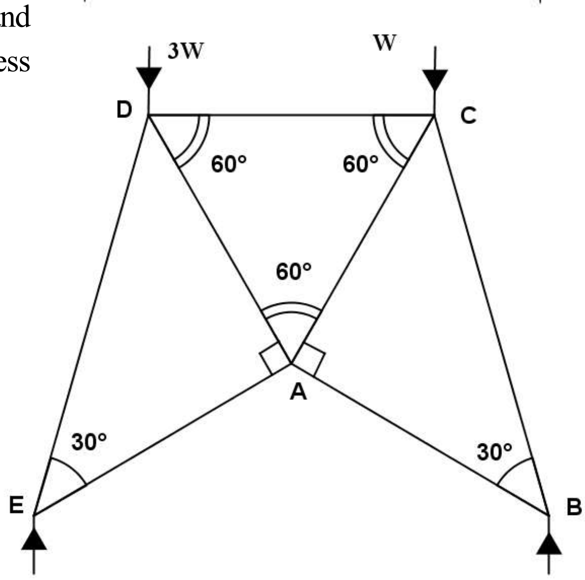
Five light rods are freely jointed to form the framework shown in the above figure. The framework is in equilibrium in a vertical plane with joint A freely hinged to a fixed point. AB is vertical, BC is horizontal, $A\hat{D}B = 90°$ and $D\hat{B}C = D\hat{C}B = 30°$. A load of 100 N hangs at C and a horizontal force P acts at B in the direction of CB.

Find P and obtain the horizontal and vertical components of the reaction on the hinge at A. Draw a stress diagram for the framework using Bow's notation. Hence determine the stresses in all five rods distinguishing tensions and thrusts.

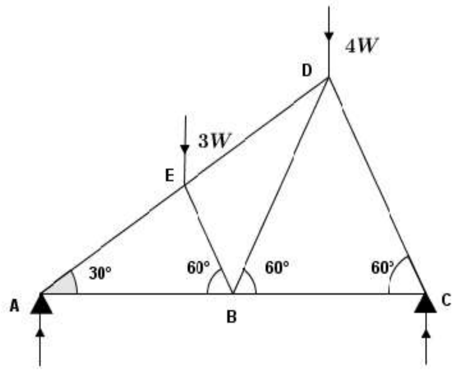
11. The given framework consists freely jointed eight light rods at A, B, C, D and E. The joints A and B are on vertical supports P at each joint. The framework carries equal loads of 100 kg at points C and D. AB is horizontal and AE=BE=AD=BC. Find the value of P. Assuming the thrust in C as X kg draw a stress diagram for the framework. If the tension on AB is Y kg, using the geometry of the stress diagram, prove that $y = 100 - (\sqrt{3} - 1)X$. Explain why the real values of x and y cannot be calculated simultaneously. Find the stress in every rod if X = Y.



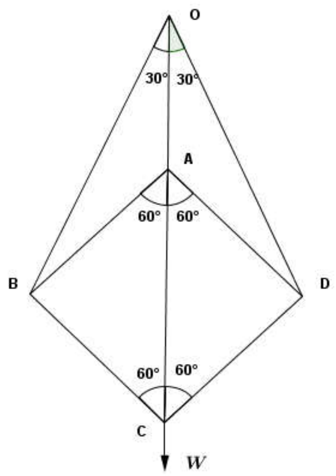
12. The given figure represents a framework which is formed by seven light rods. Ends A, B, C, D, E are freely jointed. This framework carries loads W and $2W$ at joints C and D, and is supported at B and E such that BE is horizontal. Draw a stress diagram using Bow's notation and find the stress in every rod distinguishing tensions and thrusts.



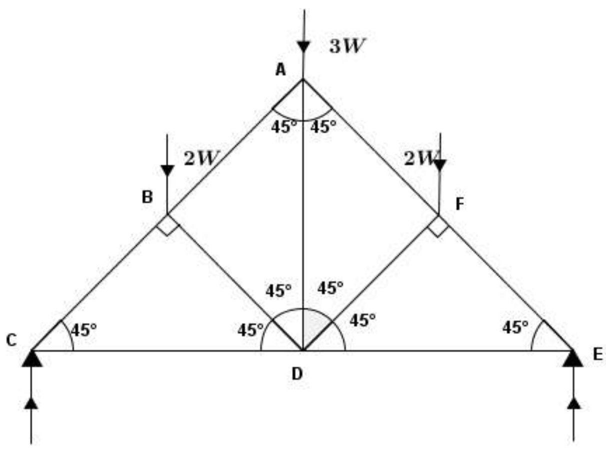
13. The framework consisting seven freely jointed light rods is placed on two supports at A and C such as the framework carries loads $4W$ and W at D and E respectively. Find the reactions at A and C. Find the stresses in each and every rod using a stress diagram, distinguishing between them compressions and tensions.



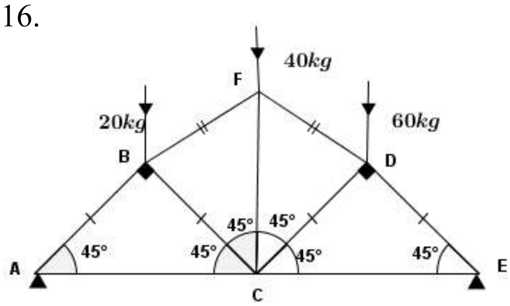
14. The given framework is a rhombus formed by freely jointed five light rods. It is hung from A by means of equal strings OB, OD, and OA is a vertical rod freely jointed at A. The diagonal AC of the rhombus is vertical and $A\hat{B}C = B\hat{O}D = 60^\circ$. When C carries a load W, find the stresses in each rod and also the tensions in the strings by using a stress diagram. Name the rods which are under tension.



15. The figure shows a framework formed by freely jointed light bars. DA is vertical. The framework is supported at C and E. It carries loads $3W$, $3W$ and W at joints A, B and F, respectively. Find the reactions at C and E. Draw a stress diagram using Bow's notation and find the stresses in each rod. Distinguish thrusts and tensions.

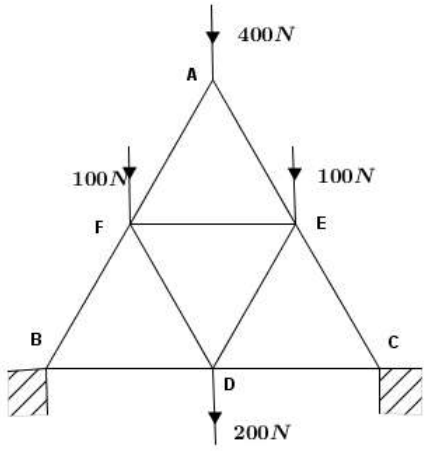


16. The given figure represents a framework of light bars loaded at joints B, F, D as indicated. The bars AC and CE are horizontal and each equal to 10 m and CF = 8 m. Also the lengths AB = BC = CD = DE and BF = FD. The frame rests on two smooth pegs at A and E. Calculate the reactions at A and E assuming that they are vertical. Draw a stress and find the stresses in the rods distinguishing between tensions and thrusts.

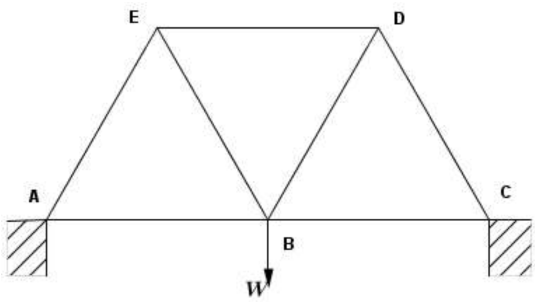


17. The given framework formed by nine equal light bars, carries loads as shown in the figure. The framework is at rest on B and C on supports such that the system is in a vertical plane.

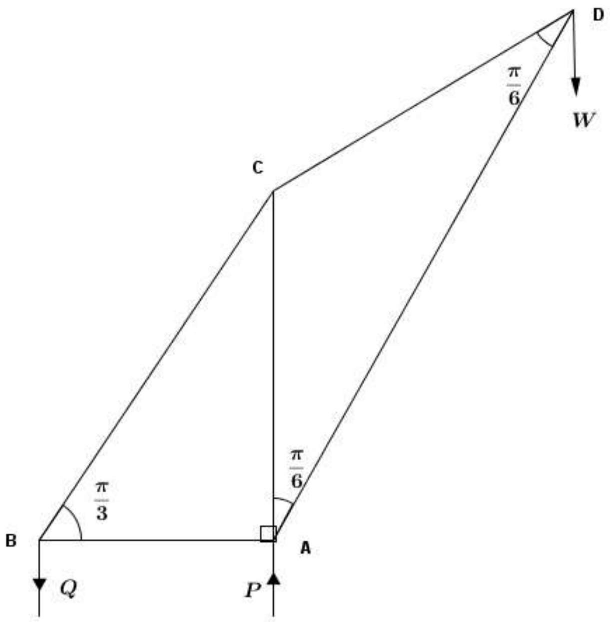
Find the reaction at B and C. Draw a stress diagram by using Bow's notation. Hence calculate the stresses in each rod distinguishing tensions and thrusts.



18. The framework of a bridge ABCDE which is formed by seven light equal rods is shown in the figure. The joints A and C are on supports which are in same horizontal level and the framework is in a vertical plane and B carries a load W. Draw a stress diagram using Bow's notation. Hence find the stress in each rod distinguishing thrusts and tensions.



19.



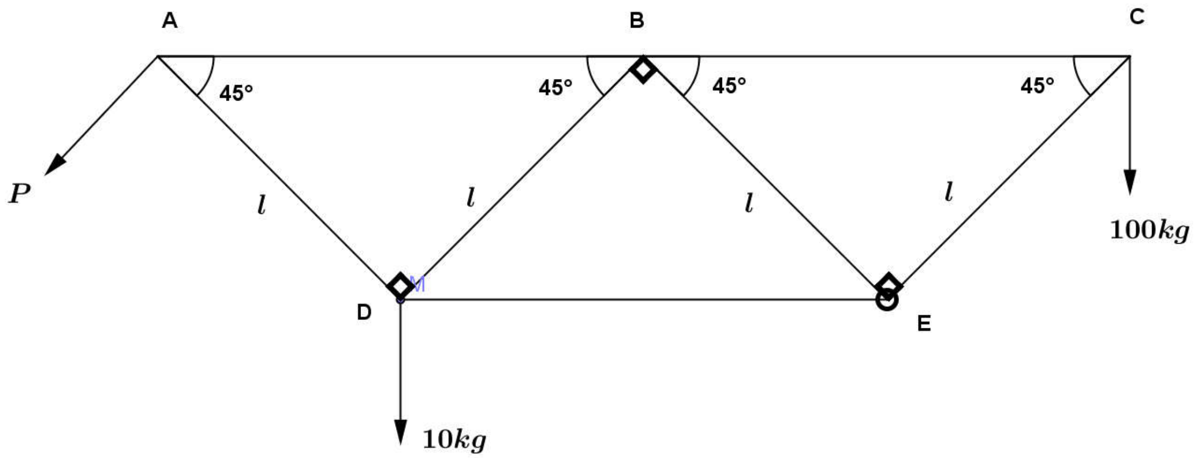
The framework consisting light rods AB, BC, CA, CD and DA are freely jointed at their extremities is placed in a vertical plane with AB horizontal and AC vertical, AB = a, $B\hat{C}D = B\hat{A}D = \frac{2\pi}{3}$ and $A\hat{B}C = \frac{\pi}{3}$. The framework supports a vertical load W at D and the equilibrium is maintained by two vertical forces P, Q at A and B respectively.

(i) Find P and Q in terms of W

(ii) Draw a stress diagram for this framework using Bow's notation.

Hence determine the stresses in the five bars distinguishing thrusts and tensions.

20.



The above framework is made by seven light rods AB, BC, AD, BD, BE, CE and DE where AD = BD = BE = CE = ℓ. The frame is hinged at E, kept in equilibrium by a force P applied at A, with loads 100 kg at C and 10 kg at D.

(i) Find the vertical and horizontal components of the reaction at E.

(ii) Find the value of P.

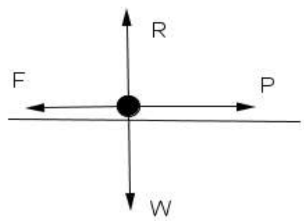
(iii) Draw a stress diagram using Bow's notation and hence find stresses in each rod distinguishing tensions and thrusts.

7.0 Friction

7.1 Introduction

When two bodies are in contact with each other the action at the point of contact of the bodies to prevent sliding of one on another is called frictional force. The frictional force on two bodies are equal in magnitude and opposite in direction.

When a horizontal force P is applied on a body, if it does not move the reason is that the force P is suppressed by an equal and opposite force. This force is called frictional force and if this force is F, then F = P.



When P is gradually increased, at some stage the body will start to move. This shows that the frictional force cannot increase beyond a limit and this is called limiting frictional force.

At limiting equilibrium,

$$\text{Coefficient of friction} = \frac{\text{Limiting frictional force}}{\text{Normal Reaction}} = \mu \text{, where } \mu \text{ is the coefficient of friction}$$

In equilibrium $\frac{F}{R} \leq \mu$

At limiting equilibrium $\frac{F_L}{R} = \mu$. (F_L - Limiting frictional force)

7.2 Laws of Friction

1. When two bodies are in contact with each other, the direction of the frictional force at the point of contact acting on the body by the other is opposite to the direction in which the body tends to move.
2. When the bodies are in equilibrium the magnitude of the frictional force is sufficient only to prevent the motion of the body. Only a certain amount of friction can be exerted called limiting friction.
3. The ratio of the limiting frictional force and the normal reaction is called the coefficient of friction and depends on the matter of which the body is composed.
4. Until the normal reaction remains unchanged, the limiting frictional force does not depend on the area and the shape of the surfaces.
5. When the motion is started, the direction of the frictional force is opposite to the direction of the motion. The frictional force after the motion is started is slightly less than the limiting frictional force before the motion.
6. The frictional force exerted by the surface on a moving body does not depend on the velocity of the body.

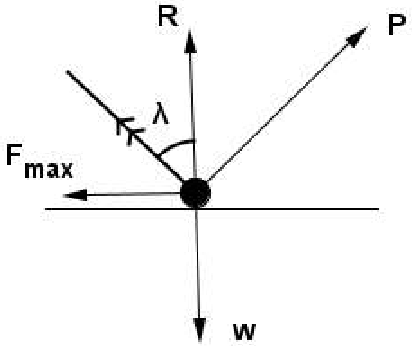
Angle of Friction

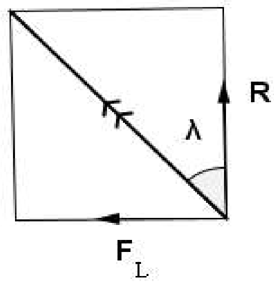
When two bodies are in contact with each other the total reaction at the point of contact is the resultant of the normal reaction and the frictional force. At limiting equilibrium, the angle λ which this resultant makes with the normal reaction is called the angle of friction.

$$\tan\lambda = \frac{F_L}{R}$$

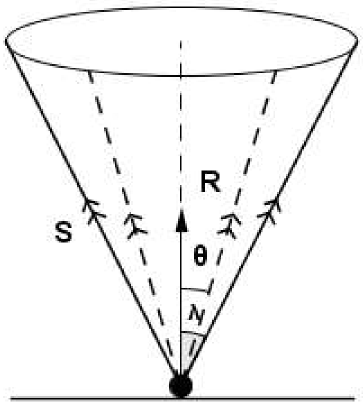
$$\frac{F_L}{R} = \mu$$

$$\tan\lambda = \mu$$





Cone of Friction



When a body is in contact with a rough surface and with the common normal at the point of contact as axes, we describe a right circular cone whose semi vertical angle is λ.

This cone is defined as cone of friction. The resultant reaction must always be within or on the surface of the cone whatever the direction the body tends to move.

- **Equilibrium of a particle on a rough horizontal surface when an external force acts**

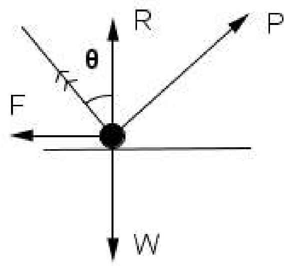
$$\frac{F}{R} = \tan\theta$$

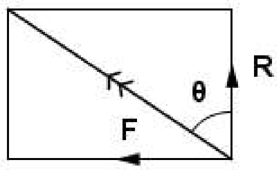
$$\frac{F}{R} \leq \mu$$

$$\tan\theta \leq \mu$$

$$\tan\theta \leq \tan\lambda$$

$$\theta \leq \lambda$$





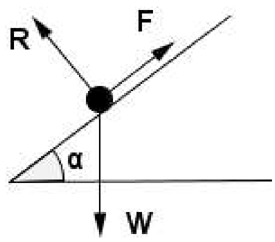
- **Equilibrium of a particle on a rough inclined plane**

Resolving parallel to the plane

$\nearrow$ $F - W\sin\alpha = 0$; $F = W\sin\alpha$

Resolving perpendicular to the plane

$\nwarrow$ $R - W\cos\alpha = 0$; $R = W\cos\alpha$



For equilibrium $\dfrac{F}{R} \leq \mu$

$$\frac{W \sin \alpha}{W \cos \alpha} \leq \tan \lambda$$

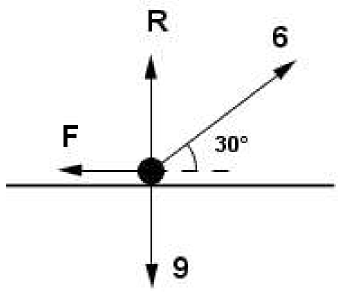
$$\tan \alpha \leq \tan \lambda$$

$$\alpha \leq \lambda$$

7.3 Worked examples

Example 1

A body of weight 9 N which is placed on a rough horizontal plane is pulled by a string inclined at an angle 30° to the horizontal. If it just begins to move when the tension in the string is 6 N, find the coefficient of friction between the body and the plane.



Resolving horizontally

$$\rightarrow \quad 6 \cos 30 - F = 0 \quad ; \quad F = 3\sqrt{3}$$

Resolving vertically

$$\uparrow \quad R + 6 \sin 30° - 9 = 0$$

$$R = 6$$

For limiting equilibrium

$$\mu = \frac{F}{R}$$

$$= \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2}$$

Example 2

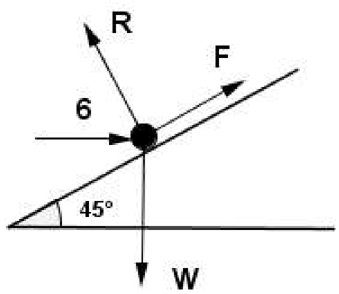
A body is placed on a plane of inclination 45° to the horizontal. The coefficient of friction between the body and the plane is $\frac{1}{3}$. A horizontal force 6 N is necessary to prevent the body from sliding down the plane.

(a) Find the weight of the body.

(b) If the motion of the body up the plane starts when the force is increased gradually find the value of the force.

(a) $\mu = \dfrac{1}{3}$

Resolving parallel to the plane



$$\nearrow \quad F + 6 \cos 45° - W \sin 45° = 0 \quad ; \quad F = \frac{W - 6}{\sqrt{2}}$$

Resolving perpendicular to the plane

$$\nwarrow \quad R - 6 \sin 45° - W \cos 45° = 0 \quad ; \quad R = \frac{W + 6}{\sqrt{2}}$$

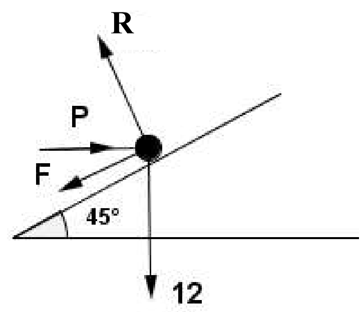
For limiting equilibrium

$$\frac{F}{R} = \mu \quad ; \quad \frac{\frac{W - 6}{\sqrt{2}}}{\frac{W + 6}{\sqrt{2}}} = \frac{1}{3}$$

$$\frac{W - 6}{W + 6} = \frac{1}{3} \quad ; \quad W = 12 \text{ N}$$

(b)

Resolving parallel to the plane



$$\swarrow \quad F - P\cos 45° + 12\sin 45° = 0 \quad ; \quad F = \frac{P - 12}{\sqrt{2}}$$

Resolving perpendicular to the plane

$$\nwarrow \quad R - P\sin 45° - 12\cos 45° = 0 \quad ; \quad R = \frac{P + 12}{\sqrt{2}}$$

In limiting equilibrium

$$\frac{F}{R} = \mu$$

$$\frac{\frac{P - 12}{\sqrt{2}}}{\frac{P + 12}{\sqrt{2}}} = \frac{1}{3}$$

$$\frac{P - 12}{P + 12} = \frac{1}{3} \quad ; \quad P = 24 \text{ N}$$

Equilibrium of a particle on a rough plane

- **The minimum force required to move a particle on a rough horizontal plane**

Let the weight of the particle be W and the angle of friction λ.

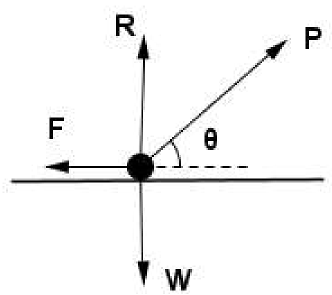
Forces acting on the particle :

(a) Weight W

(b) Frictional force F

(c) Normal reaction R

(d) Required force P at an angle θ with the horizontal



For equilibrium of the particle

Resolving horizontally

$$\rightarrow \quad P\cos\theta - F = 0 \quad ; \quad F = P\cos\theta$$

Resolving vertically

$$\uparrow \quad R + P\sin\theta - W = 0 \quad ; \quad R = W - P\sin\theta$$

For limiting equilibrium

$$\frac{F}{R} = \mu = \tan\lambda$$

$$\frac{P\cos\theta}{W - P\sin\theta} = \frac{\sin\lambda}{\cos\lambda}$$

$$P(\cos\theta\cos\lambda + \sin\theta\sin\lambda) = W\sin\lambda$$

$$P\cos(\theta-\lambda) = W\sin\lambda$$

$$P = \frac{W\sin\lambda}{\cos(\theta-\lambda)}$$

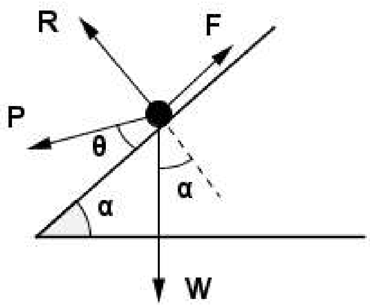
P to be minimum $\cos(\theta - \lambda) = 1$. This means $\theta = \lambda$

$$\theta = \lambda \text{ and } P_{min} = W\sin\lambda$$

- **When the inclination of the plane is less than the angle of friction, the least force required to move the particle down the plane**

Let α be the inclination of the plane. Since $\alpha < \lambda$, the particle will be in equilibrium.

Let the force applied be P at an angle θ with the plane



For equilibrium of the particle,

Resolving parallel to the plane,

$\swarrow$ $P\cos\theta + W\sin\alpha - F = 0$

Resolving perpendicular to the plane,

$\nwarrow$ $R - W\cos\alpha + P\sin\theta = 0$

At limiting equilibrium

$$\frac{F}{R} = \mu = \tan\lambda$$

$$\frac{P\cos\theta + W\sin\alpha}{W\cos\alpha - P\sin\theta} = \frac{\sin\lambda}{\cos\lambda}$$

$$P(\cos\theta\cos\lambda + \sin\theta\sin\lambda) = W(\sin\lambda\cos\alpha - \cos\lambda\sin\alpha)$$

$$P\cos(\theta-\lambda) = W\sin(\lambda-\alpha)$$

$$P = \frac{W\sin(\lambda-\alpha)}{\cos(\theta-\lambda)}$$

P to be minimum $\cos(\theta-\lambda) = 1$;

i.e. $\theta = \lambda$ and the least value of $P = W\sin(\lambda-\alpha)$

- **When the inclination of the plane is less than the angle of friction, the least force required to move the particle up the plane**

Let the inclination be α. Since $\alpha < \lambda$, the particle will be in equilibrium.

Let the force applied be P at an angle θ with the plane.

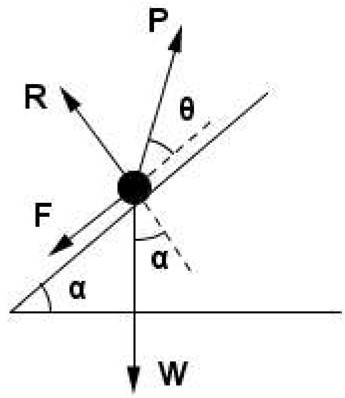
For equilibrium,

Resolving parallel to the plane,

$$\nearrow \; P\cos\theta - F - W\sin\alpha = 0$$

Resolving perpendicular to the plane,

$$\nwarrow \; R + P\sin\theta - W\cos\alpha = 0$$



At limiting equilibrium

$$\frac{F}{R} = \mu = \tan\lambda$$

$$\frac{P\cos\theta - W\sin\alpha}{W\cos\alpha - P\sin\theta} = \frac{\sin\lambda}{\cos\lambda}$$

$$P(\cos\theta\cos\lambda + \sin\theta\sin\lambda) = W(\sin\alpha\cos\lambda + \cos\alpha\sin\lambda)$$

$$P\cos(\theta - \lambda) = W\sin(\alpha+\lambda)$$

$$P = \frac{W\sin(\alpha+\lambda)}{\cos(\theta-\lambda)}$$

P to be minimum $\cos(\theta - \lambda) = 1$;

i.e. $\theta = \lambda$ and the least value of $P = W\sin(\alpha+\lambda)$

- **When the inclination of the plane is greater than the angle of friction, the least force required to move the particle upwards on the plane**

Since $\alpha > \lambda$, the particle will slide down on the plane.

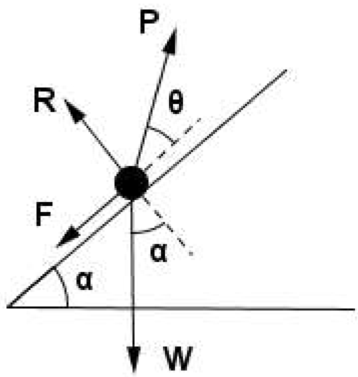
For equilibrium,

Resolving parallel to the plane,

$$\nearrow \; P\cos\theta - F - W\sin\alpha = 0$$

Resolving perpendicular to the plane,

$$\nwarrow \; R + P\sin\theta - W\cos\alpha = 0$$



At limiting equilibrium

$$\frac{F}{R} = \mu = \tan\lambda$$

$$\frac{P\cos\theta - W\sin\alpha}{W\cos\alpha - P\sin\theta} = \frac{\sin\lambda}{\cos\lambda}$$

$$P(\cos\theta\cos\lambda + \sin\theta\sin\lambda) = W(\sin\alpha\cos\lambda + \cos\alpha\sin\lambda)$$

$$P\cos(\theta - \lambda) = W\sin(\alpha+\lambda)$$

$$P = \frac{W\sin(\alpha+\lambda)}{\cos(\theta-\lambda)}$$

P to be minimum $\cos(\theta - \lambda) = 1$;

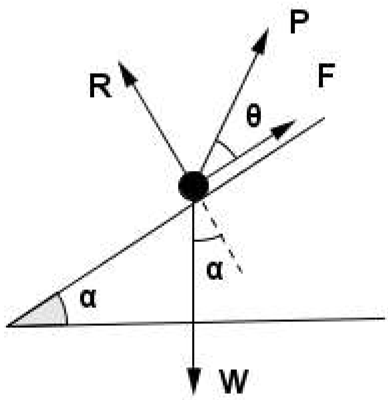
i.e. $\theta = \lambda$ and the least value of $P = W\sin(\alpha+\lambda)$

- **When the inclination of the plane is greater than the angle of friction, the least force required to support the particle**

Let α be the inclination of the plane to the horizontal. Since $\alpha > \lambda$, the particle will slide down on the plane. We have to find the least force to support.

The particle is on the point of moving down the plane.Therefore the frictional force F acts up the plane,

For equilibrium of the particle,



Resolving parallel to the plane,

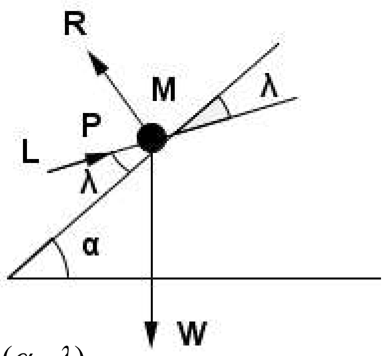
$\nearrow$ $F + P\cos\theta - W\sin\alpha = 0$

Resolving perpendicular to the plane,

$\nwarrow$ $R + P\sin\theta - W\cos\alpha = 0$

At limiting equilibrium, $\dfrac{F}{R} = m = \tan\lambda$

$$\frac{W\sin\alpha - P\cos\theta}{P\cos\alpha - W\sin\theta} = \frac{\sin\lambda}{\cos\lambda}$$



$$W(\sin\alpha\cos\lambda - \cos\alpha\sin\lambda) = P(\cos\theta\cos\lambda - \sin\theta\sin\lambda)$$

$$P\cos(\theta+\lambda) = W\sin(\alpha-\lambda)$$

$$P = \frac{W\sin(\alpha-\lambda)}{\cos(\theta+\lambda)}$$

P to be minimum $\cos(\theta+\lambda) = 1$;

i.e. $\theta = -\lambda$ and the least value of $P = W\sin(\alpha-\lambda)$

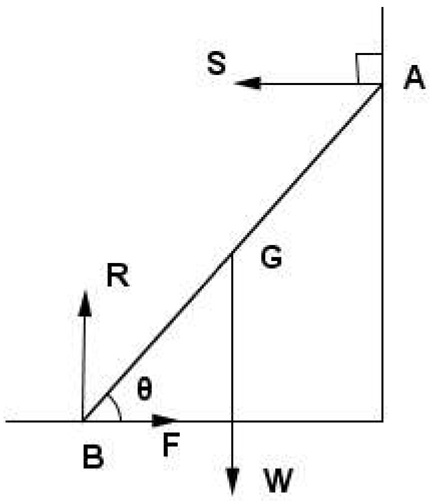
$\theta = -\lambda$ means P acts along LM and the least value of P is $W\sin(\alpha-\lambda)$

Equilibrium of rigid bodies on rough planes

Example 3

A uniform rod of length $2a$ and weight W rests one end against a smooth wall and the other end on a rough horizontal floor, the coefficient of friction being μ. If the rod is on the point of slipping show that inclination of the rod to the horizontal is $\tan^{-1}\left(\frac{1}{2}\cot\lambda\right)$ and find the reaction at the wall and on the ground, where λ is the angle of friction.

Method I



Let θ be the angle the rod makes with the horizontal.

For equilibrium of the rod AB,

Resolving horizontally,

$\rightarrow F - S = 0$; $F = S$ ------- ①

Resolving vertically,

$\uparrow R - W = 0$; $R = W$ -------- ②

Taking moments about B

$$B \quad = 0, \qquad S.2a\sin\theta - Wa\cos\theta = 0$$

$$S = \frac{W}{2}\cot\theta \quad \text{---------} \ ③$$

$$\text{From ① and ③, } F = S = \frac{W}{2}\cot\theta$$

At limiting equilibrium

$$\frac{F}{R} = \mu$$

$$\frac{W\cot\theta}{2} \times \frac{1}{W} = \tan\lambda$$

$$\cot\theta = 2\tan\lambda$$

$$\tan\theta = \frac{1}{2}\cot\lambda$$

$$\theta = \tan^{-1}\left(\frac{1}{2}\cot\lambda\right)$$

$$S = \frac{W}{2}.2\tan\lambda$$

$$= W\tan\lambda$$

Method II

The reaction S at A and weight W of the rod meet at O.

For equilibrium of the rod AB, the resultant R′, of F and R passes through O.

Since the rod is at limiting equilibrium, the angle between R and R′ is λ (angle of friction)

Applying cot rule for triangle AOB

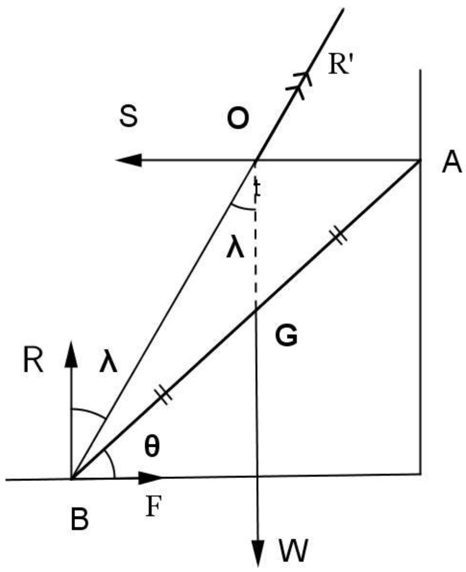
$$(BG + GA)\cot(90° - \theta) = BG\cot\lambda - GA\cot 90°$$

$$(a + a)\tan\theta = a\cot\lambda$$

$$2\tan\theta = \cot\lambda$$

$$\tan\theta = \frac{1}{2}\cot\lambda$$

$$\theta = \tan^{-1}\left(\frac{1}{2}\cot\lambda\right)$$



Reaction at the wall is $S = F = \frac{W}{2}\cot\theta$ (from ③)

$$= W\tan\lambda$$

Reaction at the ground is $= \sqrt{F^2 + R^2}$

$$= \sqrt{(W\tan\lambda)^2 + W^2}$$

$$= \sqrt{W^2(1 + \tan^2\lambda)}$$

$$= W\sec\lambda$$

Example 4

A uniform rod rests with one end on a rough ground and the other end on a rough wall. The vertical plane containing the rod is perpendicular to the wall. The coefficient of friction at the wall is μ_1 and ground is μ_2. If the rod is on the point of slipping at both ends, show that the angle the rod makes with horizontal is $\tan^{-1}\left(\dfrac{1-\mu_1\mu_2}{2\mu_2}\right)$.

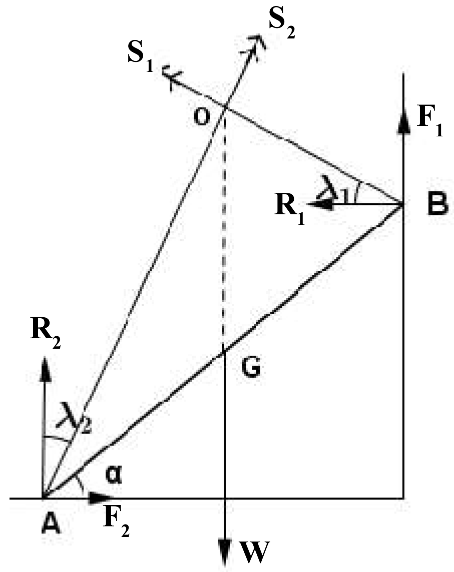
(i) The resultant of F_1 and R_1 is S_1.

(ii) The resultant of F_2 and R_2 is S_2.

(iii) Weight of the rod is W

For equilibrium of the rod the three forces S_1, S_2 and W meet at a point O.

Let $\mu_1 = \tan\lambda_1$ and $\mu_2 = \tan\lambda_2$



The angle between R_1 and S_1 is λ_1

The angle between R_2 and S_2 is λ_2

Applying Cot Rule for triangle AOB

$$(AG + GB)\cot(90° - \alpha) = AG\cot\lambda_2 - GB\cot(90° - \lambda_1)$$

$$(1+1)\tan\alpha = \frac{1}{\tan\lambda_2} - \tan\lambda_1$$

$$2\tan\alpha = \frac{1 - \tan\lambda_1\tan\lambda_2}{\tan\lambda_2}$$

$$\tan\alpha = \frac{1 - \tan\lambda_1\tan\lambda_2}{2\tan\lambda_2}$$

$$\tan\alpha = \left(\frac{1 - \mu_1\mu_2}{2\mu_2}\right)$$

$$\alpha = \tan^{-1}\left(\frac{1 - \mu_1\mu_2}{2\mu_2}\right)$$

Example 5

A uniform rod AB of weight W and length $2a$ is kept in equilibrium with the end A in contact with a rough vertical wall supported by a light inextensible string of equal length 2a connecting the other end B to a point C on the wall vertically above A. The rod is inclined at an angle θ to the upward vertical and lies in a vertical plane perpendicular to the wall.

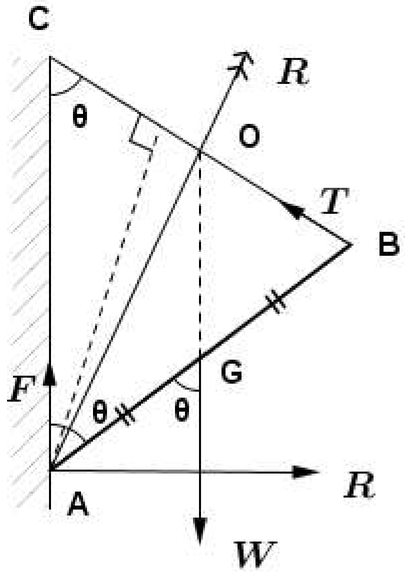
Find the tension in the string and show that $\theta \geq \cot^{-1}\left(\dfrac{\mu}{3}\right)$, where μ is the coefficient of friction.

The tension T in the string at B and the weight of the rod W meet at O.

Therefore, for equilibrium of the rod the resultant R^1 of F and R at A passes through O.

$C\hat{A}B = \theta$, since $BA = BC$, $B\hat{A}C = B\hat{C}A = \theta$

$$\therefore \; A\hat{B}C = 180 - 2\theta$$



For equilibrium of AB,

Am $= 0$

$$T.\,AB \sin(180° - 2\theta) - W.AG \sin\theta = 0$$

$$T.2a \sin 2\theta = W.a \sin\theta$$

$$T = \frac{W}{4\cos\theta}$$

$$= \frac{W \sec\theta}{4}$$

For equilibrium of AB,

Resolving horizontally,

$$\rightarrow R - T\cos(90° - \theta) = 0$$

$$R = T\sin\theta = \frac{W\tan\theta}{4}$$

Resolving vertically,

$$\uparrow \; T\cos\theta + F - W = 0$$

$$F = W - T\cos\theta$$

$$= W - \frac{W}{4} = \frac{3W}{4}$$

For equilibrium,

$$\frac{F}{R} \le \mu$$

$$\frac{3W}{4} \times \frac{4}{W\tan\theta} \le \mu$$

$$3\cot\theta \le \mu$$

$$\cot\theta \le \frac{\mu}{3}$$

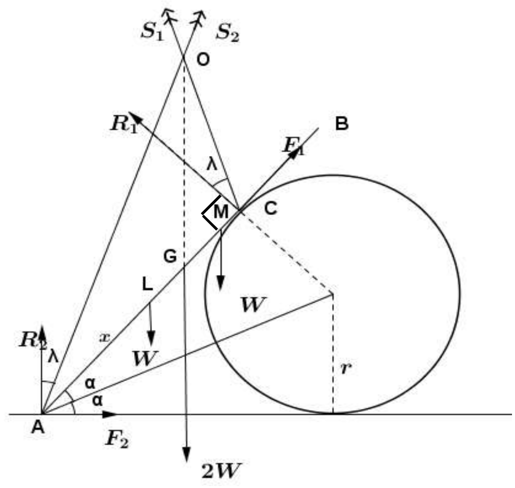
$$\theta \ge \cot^{-1}\left(\frac{\mu}{3}\right)$$

Example 6

A ladder whose centre of gravity is at a distance b from the foot, stands on a rough horizontal ground and leans in equilibrium against a rough cylindrical pipe of radius r fixed on the ground. The ladder projects beyond the point of contact with the pipe and is perpendicular to the axis of the pipe. Let λ be the angle of friction at both points where friction acts, and 2α (such that $b < \cot\alpha$), be the inclination of the ladder to the horizontal. A load of weight equal to that of the ladder is suspended from a point at a distance x measured along the ladder from its foot. The ladder is in limiting equilibrium at both points where friction acts. Show that $(b + x)\sin^2\alpha\cos 2\alpha = r\sin\alpha\cos\alpha$.

The resultant S_1 of the forces F_1 and R_1 at C,

The resultant S_2 of the forces F_2 and R_2 at A and the resulta weight of the ladder W and the weight W at meet at O.



Since equilibrium is limiting

(i) the angle between R_1 and S_1 is λ

(ii) the angle between R_2 and S_2 is λ

$AM = b, \quad AC = r\cot\alpha$

$AL = x \quad AM = b$,

Therefore $AG = AL + LG = x + \dfrac{b-x}{2} = \dfrac{b+x}{2}$

Now $AG = \dfrac{b+x}{2}$ and $GC = r\cot\alpha - \left(\dfrac{b+x}{2}\right)$

Appling Cot Rule for the triangle ACO,

$$(AG+GC)\cot(90^\circ - 2\alpha) = GC\cot\left[90^\circ - (\lambda + 2\alpha)\right] - AG\cot(90^\circ + \lambda)$$

$$AC\tan 2\alpha = GC\tan(\lambda + 2\alpha) + AG\tan\lambda$$

$$r\cot\alpha . \tan 2\alpha = \left[r\cot\alpha - \left(\frac{b+x}{2}\right)\right]\tan(\lambda+2\alpha) + \left(\frac{b+x}{2}\right)\tan\lambda$$

$$r\cot\alpha\left[\tan 2\alpha - \tan(\lambda+2\alpha)\right] = \left(\frac{b+x}{2}\right)\left[\tan\lambda - \tan(\lambda+2\alpha)\right]$$

$$r\cot\alpha\left[\frac{\sin 2\alpha}{\cos 2\alpha} - \frac{\sin(\lambda+2\alpha)}{\cos(\lambda+2\alpha)}\right] = \left(\frac{b+x}{2}\right)\left[\frac{\sin\lambda}{\cos\lambda} - \frac{\sin(\lambda+2\alpha)}{\cos(\lambda+2\alpha)}\right]$$

$$r\frac{\cos\alpha}{\sin\alpha}\left[\frac{\sin\left[2\alpha-(\lambda+2\alpha)\right]}{\cos 2\alpha . \cos(\lambda+2\alpha)}\right] = \left(\frac{b+x}{2}\right)\left[\frac{\sin\left[\lambda-(\lambda+2\alpha)\right]}{\cos\lambda . \cos(\lambda+2\alpha)}\right]$$

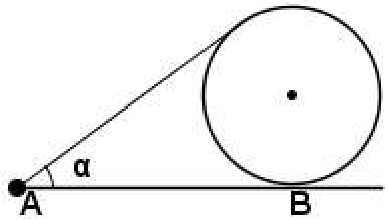
$$r\frac{\cos\alpha}{\sin\alpha}\times\frac{\sin(-\lambda)}{\cos 2\alpha} = \left(\frac{b+x}{2}\right)\frac{\sin(-2\alpha)}{\cos\lambda}$$

$$\frac{r\cos\alpha . \sin\lambda}{\sin\alpha . \cos 2\alpha} = \left(\frac{b+x}{2}\right)\frac{2\sin\alpha\cos\alpha}{\cos\lambda}$$

$$r\sin\lambda\cos\lambda = (b+x)\sin^2\alpha . \cos 2\alpha$$

Example 7

A particle A of weight w, resting on a rough horizontal floor is attached to one end of a light inextensible string wound round a right circular cylinder of radius a and weight W, that rests on the floor, touching it along a generator through a point B. The other end of the string is fastened to the cylinder. The vertical plane through the string is perpendicular to the axis of the cylinder, passes through the centre of gravity of the cylinder and intersects the floor along AB, as shown in the figure.



The string is just taut and makes an angle α with AB. The floor is rough enough to prevent the cylinder from moving at B. A couple of moment G is applied to the cylinder so that the particle is in limiting equilibrium. If μ is the coefficient of friction between the particle and floor, show that the tension in the string is $\dfrac{\mu W}{(\cos\alpha + \mu\sin\alpha)}$.

By taking moments about B, find the value of G.

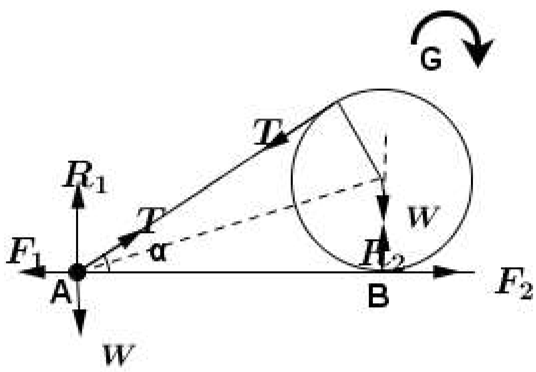
For equilibrium of the system,

Resolving horizontally

$$\rightarrow\ F_2 - F_1 = 0$$
$$F_2 = F_1$$

Resolving vertically

$$\uparrow\ R_1 + R_2 - W - w = 0$$
$$R_1 + R_2 = W + w$$



For equilibrium of the particle,

Resolving horizontally,

$$\rightarrow T\cos\alpha - F_1 = 0 \quad ; \quad F_1 = T\cos\alpha$$

Resolving vertically,

$$\uparrow\ R_1 + T\sin\alpha - w = 0\ ;\ R_1 = w - T\sin\alpha$$

At limiting equilibrium,

$$\frac{F_1}{R_1} = \mu$$

$$\frac{T\cos\alpha}{w - T\sin\alpha} = \mu \quad ; \quad T(\cos\alpha + \mu\sin\alpha) = \mu w$$

$$T = \frac{\mu w}{\cos\alpha + \mu\sin\alpha}$$

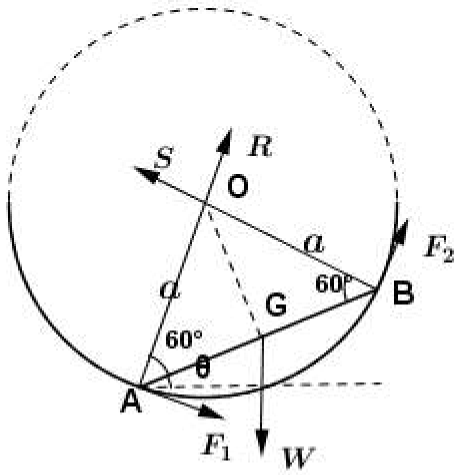
For equilibrium of the cylinder,

Bm $\quad T(a + a\cos\alpha) - G = 0$

$$G = T.\ a\,(1 + \cos\alpha)$$
$$= \frac{\mu wa\,(1+\cos\alpha)}{\cos\alpha + \mu\sin\alpha}$$

Example 8

A uniform rod of length a and weight W rests in a vertical plane inside a fixed rough hemispherical bowl of radius a. The rod is in limiting equilibrium inclined at an angle θ to the horizontal, and the coefficient of friction is $\mu\left(<\sqrt{3}\right)$. Show that the reaction at the lower end of the rod is $\dfrac{W\cos\theta}{\sqrt{3}-\mu}$ and find the reaction at the upper end. Hence show that $\tan\theta = \dfrac{4\mu}{3-\mu^2}$.



Since ther rod is in limiting equilibrium,

$F_1 = \mu R$ and $F_2 = \mu S$

For equilibrium of AB, Taking moment about B

$$B \curvearrowleft \quad -R.\,a\sin 60° + \mu R\,.\,a\sin 30° + w.\frac{a}{2}\cos\theta = 0$$

$$-R\frac{\sqrt{3}}{2} + \mu R.\frac{1}{2} + w.\frac{1}{2}\cos\theta = 0$$

$$R\left(\sqrt{3}-\mu\right) = w\cos\theta \quad ; \quad R = \frac{w\cos\theta}{\sqrt{3}-\mu} \text{ ---------(1)}$$

Taking moment about A

$$A \curvearrowleft \quad S.a\sin 60° + \mu S.a\sin 30° - w.\frac{a}{2}\cos\theta = 0$$

$$\frac{\sqrt{3}S}{2} + \frac{\mu S}{2} = \frac{w\cos\theta}{2}$$

$$S = \frac{w\cos\theta}{\left(\sqrt{3}+\mu\right)} \text{ ---------------(2)}$$

Taking moment about O

$$O \curvearrowleft \quad F_1.a + F_2.a - w\left(\frac{a}{2}\cos\theta - a\cos(60+\theta)\right) = 0$$

$$\mu(R+S) = w\left(\frac{1}{2}\cos\theta - \frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\right)$$

$$\mu\left[\frac{w\cos\theta}{\sqrt{3}-\mu} + \frac{w\cos\theta}{\sqrt{3}+\mu}\right] = \frac{\sqrt{3}}{2}w\sin\theta$$

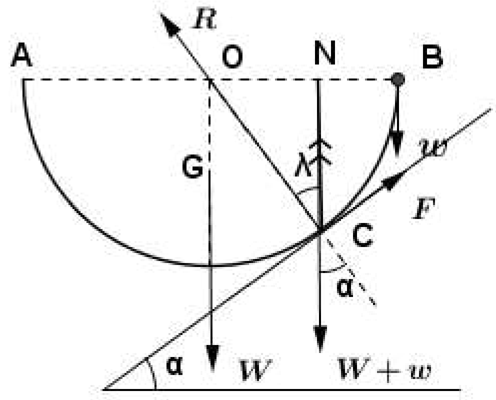
$$\frac{\mu\cos\theta\times 2\sqrt{3}}{3-\mu^2} = \frac{\sqrt{3}}{2}w\sin\theta$$

$$\tan\theta = \frac{4\mu}{3-\mu^2}$$

Example 9

A uniform solid hemisphere of weight W is placed with its curved surface on a rough plane inclined at an angle α to the horizontal. When a small weight w is attached to a point on the circumference of its plane surface, the plane surface becomes horizontal. Show that if μ is the coefficient of friction, then

$$\mu = \frac{w}{\sqrt{W(W+2w)}} = \tan\alpha .$$



Centre of gravity of the hemisphere is at G and OG $= \frac{3}{8}a$.

The forces F and R act at C on the hemisphere.

The resultant of W and w also should pass through C.

Taking moments about N,

$$W.\text{ON} - w.\ \text{BN} = 0$$

$$W.\text{ON} = w\,(a - \text{ON})$$

$$(W - w)\,.\,\text{ON} = w.a$$

$$\text{ON} = \frac{w.a}{W + w}$$

For equilibrium of the system the resultant of F and R must be equal to $(W+w)$ in magnitude and opposite in direction.

Since the equilibrium is limiting, $\text{O}\hat{\text{C}}\text{N} = \lambda$

$$\tan\lambda = \frac{\text{ON}}{\text{CN}} = \frac{\text{ON}}{\sqrt{a^2 - \text{ON}^2}}$$

$$= \frac{\frac{w.a}{W+w}}{\sqrt{a^2 - \frac{w^2a^2}{(W+w)^2}}}$$

$$= \frac{w}{\sqrt{W^2 + 2Ww}}$$

$$\mu = \frac{w}{\sqrt{W(W+2w)}}$$

$$= \tan\alpha \quad (\text{since } \lambda = \alpha)$$

Example 10

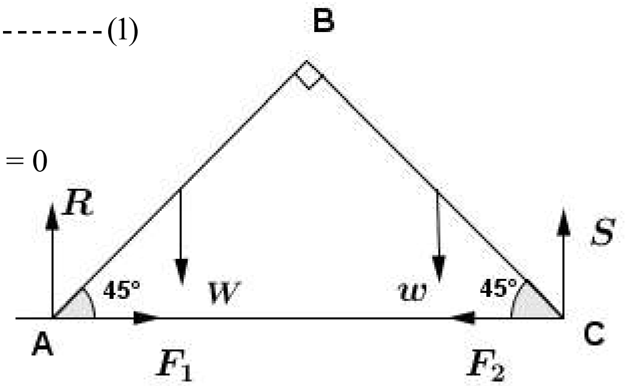
Two uniform rods AB and BC of equal length and of weights W and w ($W > w$) respectively are freely jointed at B. The rods rest in equilibrium in a vertical plane with $A\hat{B}C = \frac{\pi}{2}$ and the ends A and C on a rough horizontal ground. If μ is the coefficient of friction between the rods and the ground, show that the least possible value of μ is $\frac{W+w}{W+3w}$ in order to preserve the equilibrium. If $\mu = \frac{W+w}{W+3w}$, prove that the slipping is about to occur at C but not at A.

For equilibrium of the system,

$\rightarrow \quad F_1 - F_2 = 0 \quad ; \quad F_1 = F_2 \quad (= F, \text{say})$

$\uparrow \quad R + S - W - w = 0$

$R + S = W + w \qquad \text{-----------------(1)}$



$A \circlearrowleft = 0$

$S.\, 4a \cos 45° - w.\, 3a \cos 45° - Wa \cos 45° = 0$

$S = \frac{W+3w}{4} \quad \text{and} \quad R = \frac{3W+w}{4}$

For equilibrium of AB, $B \circlearrowleft = 0$

$$F_1 .2a \sin 45° + Wa \cos 45° - R.2a \cos 45° = 0$$

$$2F_1 + W - 2R = 0$$

$$F_1 = R - \frac{W}{2}$$

$$= \frac{3W+w}{4} - \frac{W}{2}$$

$$= \frac{W+w}{4}$$

$$F_1 = F_2 = F = \frac{W+w}{4}$$

For equilibrium of the system

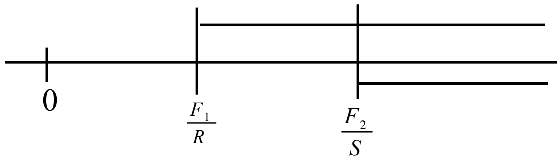
$$\frac{F_1}{R} \leq \mu \quad , \quad \frac{F_2}{S} \leq \mu$$

$$\frac{F_1}{R} = \frac{\frac{W+w}{4}}{\frac{3W+w}{4}} = \frac{W+w}{3W+w} \leq \mu$$

$$\frac{F_2}{S} = \frac{\frac{W+w}{4}}{\frac{W+3w}{4}} = \frac{W+w}{W+3w} \leq \mu$$

Now, $\quad R - S = \dfrac{3W + w}{4} - \dfrac{W + 3w}{4} = \dfrac{W - w}{2} > 0$

i.e. $\quad R > S$



$$R > S\ (>0) \Rightarrow \frac{1}{R} < \frac{1}{S}$$

$$\Rightarrow \frac{F}{R} < \frac{F}{S}$$

$$\Rightarrow \frac{F_1}{R} < \frac{F_2}{S}$$

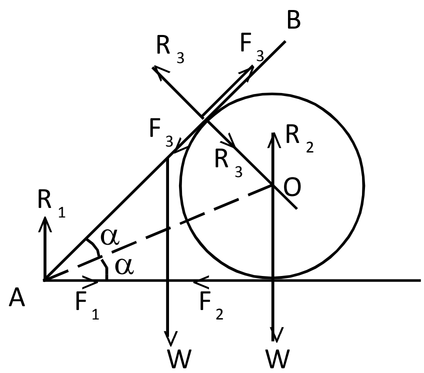
For equilibrium to be possible, $\dfrac{F_1}{R} \leq \mu$, $\dfrac{F_2}{S} \leq \mu$

The least possible value is $\dfrac{F_2}{S} = \dfrac{W + w}{W + 3w}$

If $\mu = \dfrac{W + w}{W + 3w}$, slipping first occurs at C

Example 11

A uniform plank AB of length 4ℓ and weight W rests with one end A on level ground and leans against a cylinder of radius ℓ such that the point of contact between the plank and the cylinder is at a distance 3ℓ from A. The cylinder is uniform and of weight W and rests on the ground with its axis perpendicular to the vertical plane containing the plank. Find the frictional force at each point of contact and show that for equilibrium to be possible $\mu \geq \dfrac{8}{21}$, where μ is the coefficient of friction.



For equilibrium of the system,

Resolving horizontally

$\rightarrow F_1 - F_2 = 0 \quad ; \quad F_1 = F_2$

Resolving vertically,

$\uparrow R_1 + R_2 - 2W = 0$

$R_1 + R_2 = 2W$ .... ①

For equilibrium of the sphere,

Om $\quad F_2.a - F_3.a = 0 \ ; \ F_2 = F_3$

Hence $F_1 = F_2 = F_3$ .... ②

For equilibrium of the rod AB,

Am $\quad R_3.3\ell - W.2\ell\cos 2\alpha = 0$

$R_3 = \dfrac{2W\cos 2\alpha}{3} = \dfrac{8W}{15}$ .... ③

For equilibrium of the system,

Am $\quad R_2.3\ell - W.3\ell - W.2\ell\cos 2\alpha = 0$

$3R_2 = 3W + 2W \times \dfrac{4}{5}$

$R_2 = \dfrac{23W}{15}$; From ① $R_1 = \dfrac{7W}{15}$ ....④

For equilibrium of AB,

Resolving along AB,

$$\nearrow F_3 + F_1 \cos 2\alpha + R_1 \sin 2\alpha - W \sin 2\alpha = 0$$

$$F_3 + F_1 \cos 2\alpha = \left(W - \frac{7W}{15}\right)\sin 2\alpha$$

$$F_1\left(1+\cos 2\alpha\right) = \frac{8W}{15}\times\frac{3}{5} = \frac{24W}{75} \quad (\text{since } F_1 = F_3)$$

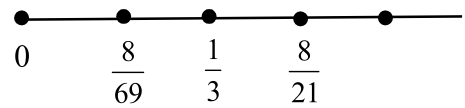
$$F_1\left(1+\frac{4}{5}\right) = \frac{24W}{75}; \quad F_1 = \frac{8W}{45}$$

For equilibrium to be possible,

$$\frac{F_1}{R_1} \le \mu\ ; \quad \frac{F_2}{R_2} \le \mu, \frac{F_3}{R_3} \le \mu$$

$$\frac{8W}{45}\times\frac{15}{7W} \le \mu\ ; \quad \frac{8W}{45}\times\frac{15}{23W} \le \mu; \quad \frac{8W}{45}\times\frac{15}{8W} \le \mu$$

i.e $\quad \mu \ge \frac{8}{21},\ \mu \ge \frac{8}{69}\ ;\ \mu \ge \frac{1}{3}$



Hence for equilibrium to be possible $\mu \ge \frac{8}{21}$

Example 12

An equilateral triangle ABC rests in a vertical plane with the side BC on a rough horizontal plane. A gradually increasing horizontal force is applied on its highest vertex A, in the plane of the triangle. Prove that the triangle will slide before it tilts if the coefficient of friction be less than $\frac{\sqrt{3}}{3}$.

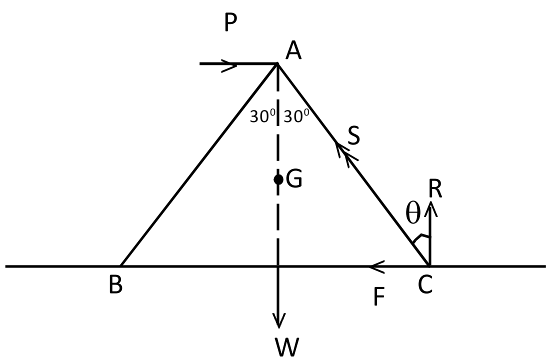
Method I

Forces acting on the triangle ABC are

(i) Weight W at G

(ii) Horizontal force P at A

(iii) Frictional force F and normal reaction at A



If the triangle topples, it topples about C.

At the point of toppling the normal reaction acts at C.

Weight W at G and the horizontal force P at A meet at A.

Therefore, the resultant S of F and R passes through A. (along CA)

Let θ be the angle between R and S.

(i) If $\lambda < \theta$, slides before toppling

(ii) If $\lambda > \theta$, topples before sliding

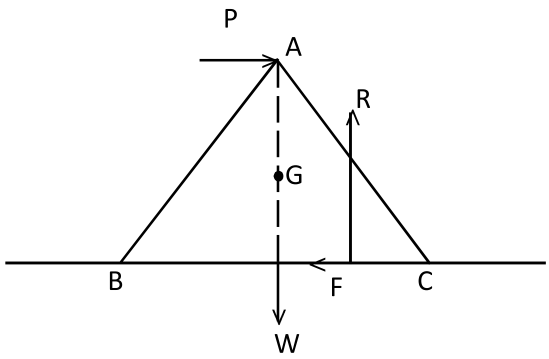
If $\lambda < \theta$, then $\tan\lambda < \tan\theta$

i.e. $\tan\lambda < \tan 30°$

$\tan\lambda < \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$; $\mu < \dfrac{\sqrt{3}}{3}$. Hence if $\mu < \dfrac{\sqrt{3}}{3}$, the triangle will slide before it topples.

Method II

For equilibrium of the triangle ABC.



Resolving horizontally,

$\rightarrow P - F = 0 \quad ; \; F = P$ .... ①

Resolving vertically

$\uparrow R - W = 0 \quad ; \; R = W$ .... ②

At limiting equilibrium

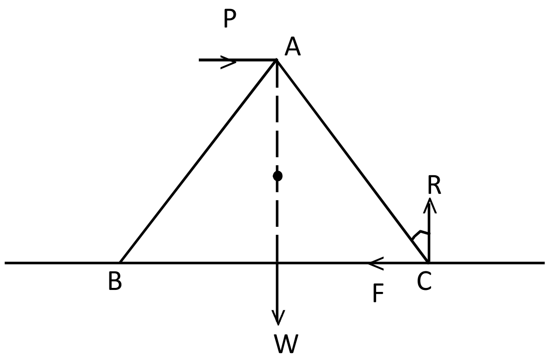
$$\frac{F}{R} = \mu \;\; ; \;\; \frac{P}{W} = \mu; \;\; P = \mu W$$

For equilibrium of the triangle ABC,

$\rightarrow P - F = 0 \quad ; \; F = P$

$\uparrow R - W = 0 \quad ; \; R = W$

At the point of toppling R will act at C



Taking moments about B

B↺ $\quad R \times 2a - P\sqrt{3}a - W.a = 0$

$$P = \frac{W}{\sqrt{3}}$$

When $P = \mu W$, lamina begins to slide.

When $P = \dfrac{W}{\sqrt{3}}$, lamina toples about C.

If $\mu W < \dfrac{W}{\sqrt{3}}$, lamina will slide before it topples.

ie If $\mu < \dfrac{1}{\sqrt{3}}$, lamina will slide before it topples.

7.4 Exercises

1. Find the least force which will move a mass of 80 kg up a rough plane inclined to the horizontal at 30°. The coefficient of friction is $\frac{3}{4}$.

2. If the least force which will move a weight up a plane of inclination α is twice the least force which will just prevent the weight from slipping down the plane, show that the coefficient of friction between the weight and the plane is $\frac{1}{3}\tan\alpha$.

3. The least force which will move a weight up an inclined plane is P. Show that the least force, acting parallel to the plane, which will move the weight upwards is $P\sqrt{1+\mu^2}$, where μ is the coefficient of friction.

4. The force P acting along a rough inclined plane is just sufficient to maintain a body on the plane, the angle of friction λ being less than α, the angle of plane. Prove that the least force, acting along the plane, sufficient to drag the body up the plane is $P\frac{\sin(\alpha+\lambda)}{\sin(\alpha-\lambda)}$.

5. A uniform ladder rests against a vertical wall at an angle 30° to the vertical. If it is just on the point of slipping down find the coefficient of friction assuming it to be the same for the wall and the ground.

6. A uniform ladder of weight w rests on a rough horizontal ground and against a smooth vertical wall inclined at an angle α to the horizontal. Prove that a man of weight W can climb up the ladder without the ladder slipping, if $\frac{w}{W} > \frac{2(1 - \mu\tan\alpha)}{2\mu\tan\alpha - 1}$

7. A straight uniform beam of length 2ℓ rests in limiting equilibrium in contact with a rough vertical wall of height h, with one end on a horizontal plane and the other end projecting beyond the wall. If both the wall and the plane are equally rough, prove that λ, the angle of friction is given by $h.\sin 2\lambda = \ell\sin\alpha\cos 2\alpha$ where α is the inclination of the beam to the horizontal.

8. A uniform ladder rests with its ends against a rough vertical wall and an equally rough horizontal ground, the coefficient of friction at both points of contacts is $\frac{1}{3}$. Prove that if the inclination of the ladder to the vertical is $\tan^{-1}\frac{1}{2}$, a weight equal to that of the ladder cannot be attached to it at a point more than $\frac{9}{10}$th of the distance from the foot of the ladder without destroying the equilibrium.

9. A heavy uniform rod of length $2a$ lies over a rough peg with one extremity leaning against a rough vertical wall. If ℓ be the distance of the peg from the wall and the point of contact of the rod with the wall is above the peg, if the rod is on the point of sliding downwards show that $\sin^3\theta = \frac{c}{a}\cos^2\lambda$ where λ is the angle of friction at both contact points and θ is the angle between the rod and downward vertical.

10. A uniform ladder of length ℓ rests on a rough horizontal ground with its upper end projecting slightly over smooth horizontal rail at a height a. If the ladder is about to slip and λ is the angle of friction on the ground, prove that $\tan\lambda = \frac{a\sqrt{\ell^2 - a^2}}{\ell^2 + a^2}$

11. A uniform rod is in limiting equilibrium, one end resting on a rough horizontal plane and the other on an equally rough plane inclined an angle α to the horizontal, λ be the angle of friction and the rod be in a vertical plane, show that the rod is inclined to be horizontal at an angle $\tan^{-1}\left[\frac{\sin(\alpha - 2\lambda)}{2\sin\lambda\sin(\alpha - \lambda)}\right]$

12. A uniform rod is placed within a fixed rough vertical circular loop. If the rod subtends an angle of 60° at the center of the loop and coefficient of friction is $\frac{1}{\sqrt{3}}$, show that in the position of limiting equilibrium the inclination of the rod to the horizontal is $\sin^{-1}\sqrt{\frac{3}{7}}$.

13. Two equal uniform rods AC, CB are freely joined at C and rests in a vertical plane with the ends A,B in contact with a rough horizontal plane. If the equilibrium is limiting and the coefficient of friction is μ. Show that $\sin A\hat{C}B = \frac{4\mu}{1+\mu^2}$.

14. A uniform lamina in the shape of an equilateral triangle rests with one vertex on a horizontal plane and the other vertex against a smooth vertical wall. The vertical plane containg the lamina is perpendicular to the wall. Show that the least angle that its edge through these vertices can make with the horizontal plane is given by $\cot\theta = 2\mu + \frac{1}{\sqrt{3}}$, μ being the coefficient of friction.

15. A uniform ladder AB of length $2a$ and weight W rests with one end A on a rough horizontal floor and the other end B against a rough vertical wall, μ being the coefficient of friction at both ends of the ladder. The ladder is in inclined to the floor at an angle $\frac{\pi}{4}$ and a small cat of weight nW gently climb up the ladder, starting from A. Show that, in the position of limiting equilibrium of the ladder, the cat has climbed a distance $\frac{a}{n(1+\mu^2)}\left[\mu^2(1+2n) + 2\mu(1+n) - 1\right]$ along the ladder.

Given further that $\mu = \frac{1}{2}$, show that the cat reach the top of the ladder before the ladder slips, if $n < \frac{1}{4}$. what happens if $n = \frac{1}{4}$

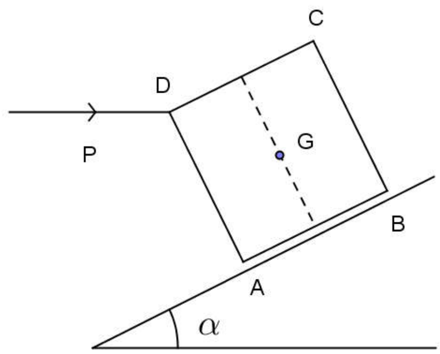
16. A uniform ladder AB of length ℓ and weight w rests with end A on a rough horizontal ground and with the other end B against a smooth vertical wall. The ladder is in a vertical plane perpendicular to the wall and is inclined an angle of to the ground. The coefficient of friction between the ladder and the ground is μ. A force P is applied horizontally towards the wall at the point C on the ladder with $AC = a(< \ell)$ so that limiting equilibrium is attained with the ladder on the point of sliding towards the wall. Show that $P = \frac{\ell w}{2(\ell - a)}(2\mu + \tan\alpha)$

17. Two uniforms rod AB, BC of equal weight but different lengths, are freely jointed together at B and placed in a vertical plane over two equally rough fixed pegs in the same horizontal line. The inclination of the rods to the horizontal are α, β and they are both on the point of slipping. Prove that the inclination θ to the horizontal of the reaction at the hinge is given by $2\tan\theta = \cot(\beta + \lambda) - \cot(\alpha - \lambda)$ where λ is the angle of friction at the pegs.

18. Two uniform equal ladders of length ℓ are hinged at the top and rest on a rough floor forming an isosceles triangle with the floor of vertical angle 2θ A man whose weight is n times that of either ladder goes slowly up one of them. Calculate the reaction at the floor when his distance from the top is *x*, and show that slipping begins when $\frac{nx}{\ell} = \frac{2\mu - \tan\theta}{\mu - \tan\theta} + n$

19. A smooth cylinder of radius a is fixed on a rough horizontal table with its axis parallel tothe table. A uniform rod ACB of length 6a and mass M rests in equilibrium with the end A on the table and the point C touching the cylinder. The vertical plane containing the rod is perpendicular to the axis of the cylinder and the rod makes an angle 2θ with the table.
    a) Show tha the magnitude of the force exerted by the cylinder on the rod is $3Mg\cos 2\theta.\tan\theta$
    b) Show also that μ, the coefficient of friction between the rod and the table, is given by $\mu(\cot\theta - 3\cos^2 2\theta) = 3\sin 2\theta \cos 2\theta$, if the equilibrium is limitimg

20. ABCD represents the central vertical cross section of a uniform cube of side $2a$ and weight W. The cube is placed on a rough plane of inclination α to the horizontal. A gradually increasing horizontal force P is applied at the point D as shown in the diagram, the coefficient od friction between the cube and the plane being μ. Find the range of value of μ so that the equilibrium is broken by moving up the plane given $\tan\alpha = \frac{1}{2}$.

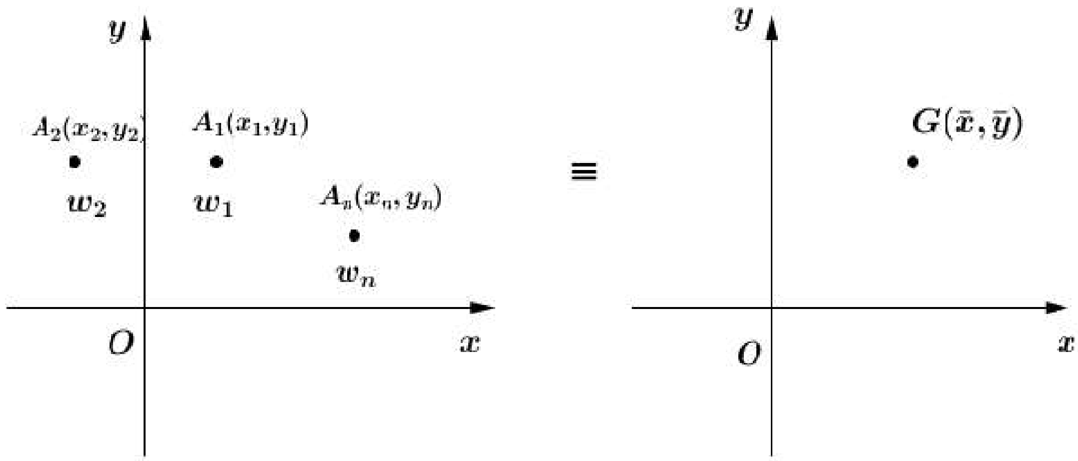


8.0 Centre of Gravity

8.1 Centre of gravity of system of particles

Centre of gravity of a body or a system of particles rigidly connected together is the point through which the line of action of the weight of the body always passes in whatever the position the body is placed.

Centre of gravity of system of particles



Let the particles of weights w_1, w_2 .... w_n be placed at points A_1, A_2, ... A_n lying in one plane. Let the coordinates of these points referred to rectangular axis OX, OY be $(x_1, y_1), (x_2, y_2)$ ....(x_n, y_n)

Let $(\bar{x}, \bar{y})$ be the coordinate of the centre of gravity referred to OXY.

Then weights of the particles form a system of parallel forces, whose resultant $(w_1 + w_2 + \ldots\ldots + w_n)$ acts at $(\bar{x}, \bar{y})$. Suppose the plane to be horizontal, and taking moments about OX and OY for the forces and the resultant, we have

$$\bar{x}(w_1 + w_2 + \ldots\ldots + w_n) = w_1x_1 + w_2x_2 + \ldots\ldots + w_nx_n$$

$$\bar{x} = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}$$

$$\bar{y}(w_1 + w_2 + \ldots\ldots + w_n) = w_1y_1 + w_2y_2 + \ldots\ldots + w_ny_n$$

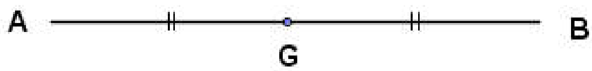
$$\bar{y} = \frac{\sum_{i=1}^{n} w_i y_i}{\sum_{i=1}^{n} w_i}$$

Note :

In uniform bodies centre of gravity, centre of mass, and the centroid are usually the same.

8.2 Centre of gravity of uniform bodies

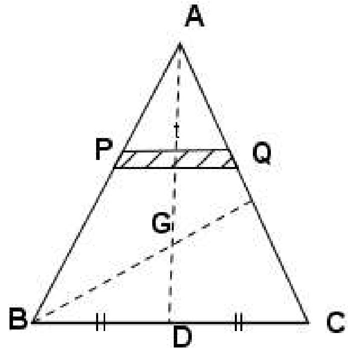
Centre of gravity of a uniform rod



AB is a uniform rod and G is the midpoint of AB.

Then G is the centre of gravity of the rod AB

Centre of gravity of a uniform triangular lamina



Let AB be a triangular lamina. Suppose it is divided into a very large number of narrow strips, such as PQ parallel to BC.

Centre of gravity of each strip is at its midpoint. Hence the centre of gravity of the whole triangle lies on the line going through the midpoints of the strips.

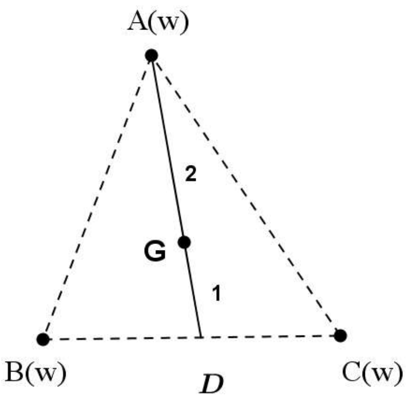
Thus the centre of gravity is in the median AD

Similarly the centre of gravity lies on the medians through B and C.

Therefore, the centre of gravity of the lamina is the point of intersection of the medians where $\frac{AG}{GD} = \frac{2}{1}$

Centre of gravity is the point on the median at a distance equal to two thirds from the vertex.

Centre of gravity of three equal particles placed at the vertices of a triangle



The weights w at B and C are equivalent to $2w$ at D where D is the midpoint of BC.

Now the system is equivalent to w at A and $2w$ at D.

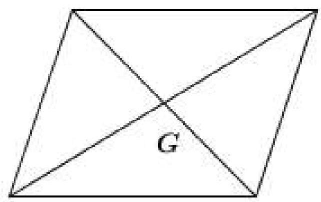
The weights w at A and $2w$ at D are equivalent to $3w$ at G.

Hence centre of gravity of this system is the intersection point of the medians.

The centre of gravity of any uniform triangular lamina is the same as that of three equal particles placed at the vertices of the triangle.

Centre of gravity of a uniform parallelogram lamina

Centre of gravity of a parallelogram lamina is the intersection point of the diagonals.



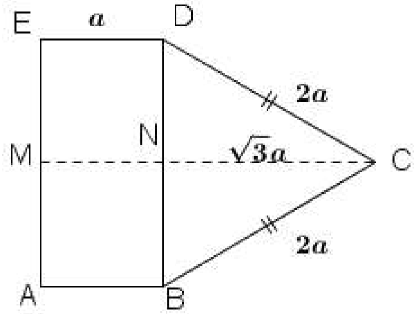
Centre of gravity of a uniform circular ring

The circular ring is symmetric about any diameter. Therefore, the centre of gravity of the circular ring is the point where the diameters meet, i.e. the centre of the circular ring.

8.3 Worked examples

Example 1

One side of a rectangle is twice of the other. On the longer side, an equilateral triangle is described. Find the centre of gravity of the lamina formed by the rectangle and the triangle.



Let AB = a, then AE = $2a$

By symmetry centre of gravity of the lamina lies on MC.

where M is the mid point of AE

Area of ABDE $= 2a^2$

Area of BCD $= \sqrt{3}a^2$

Let w be the weight of unit area.

Lamina	weight	Centre of gravity from M along MC
ABDE	$2a^2w$	$\frac{a}{2}$
BDC	$\sqrt{3}a^2w$	$a+\frac{1}{3}\sqrt{3}a$
ABCDE	$\left(2+\sqrt{3}\right)a^2w$	$\overline{x}$

Take moment about AE.

$$\text{AE} \qquad \left(2+\sqrt{3}\right)a^2w\ \overline{x} = 2aw\ \frac{a}{2}+\sqrt{3}a^2w\left(a+\frac{\sqrt{3}a}{3}\right)$$

$$\left(2+\sqrt{3}\right)\ \overline{x} = a+\sqrt{3}a+a$$

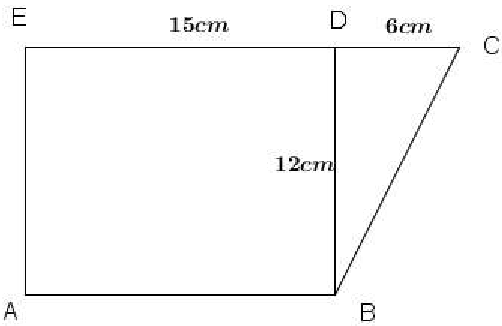
$$=\left(2+\sqrt{3}\right)a$$

$$\overline{x} = a$$

Hence centre of gravity is N , midpoint of BD

Example 2

The figure shows a uniform lamina ABCDE where ABDE is a rectangle and BCD is a right angled triangle. Find the centre of gravity of the above lamina. If this lamina is suspended from C, find the angle between CE and the vertical.



Area of ABDE = 15 x 12 = 180 cm^2

Area of BCD $= \frac{1}{2}\times 12\times 6 = 36\text{ cm}^2$

Let w be the weight per unit area

Lamina	Weight	Distance of centre of gravity	
		from AE	from AB
ABDE	$180w$	$\frac{15}{2}$ cm	6 cm
BCD	$36w$	$15+\frac{1}{3}\times 6 = 17$ cm	$\frac{2}{3}\times 12 = 8$ cm
ABCE	$216w$	$\overline{x}$	$\overline{y}$

Taking moment about AE,

$$216w\,\bar{x} = 180w \times \frac{15}{2} + 36w \times 17$$
$$12\,\bar{x} = 75 + 34$$
$$= 109$$
$$\bar{x} = \frac{109}{12}\text{ cm}$$

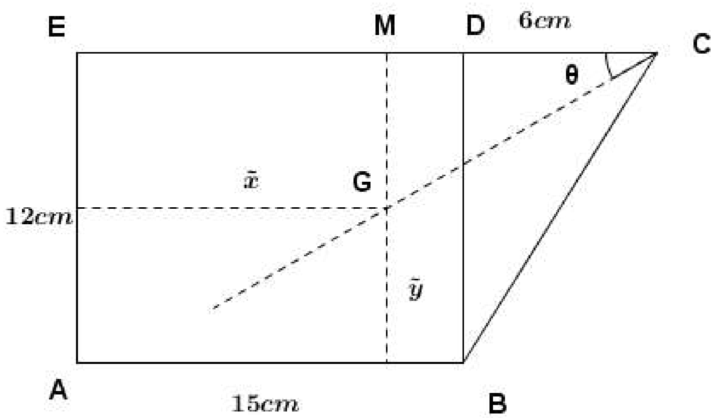
Taking moment about AB

$$216w\,\bar{y} = 180w \times 6 + 36w \times 8$$
$$12\,\bar{y} = 60 + 16$$
$$= 76$$
$$\bar{y} = \frac{19}{3}\text{ cm}$$

Centre of gravity is at the distance $\frac{19}{3}$ cm from AE and $\frac{109}{12}$ cm from AB

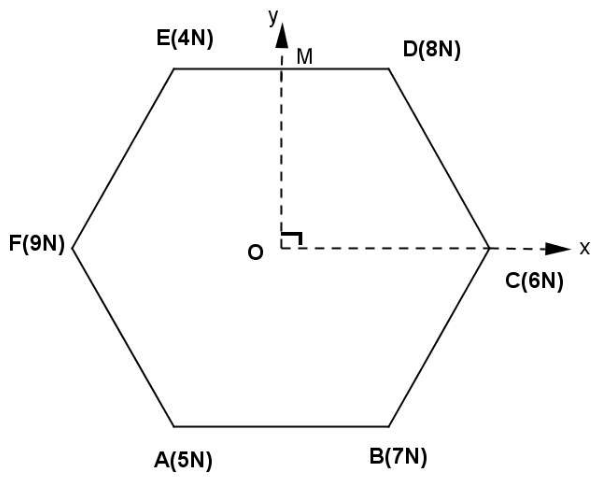
When the lamina hangs freely from C, CG is vertical.

$$\tan\theta = \frac{MG}{CM}$$
$$= \frac{12-\bar{y}}{21-\bar{x}}$$
$$= \frac{12-\frac{19}{3}}{21-\frac{109}{12}}$$
$$= \frac{68}{143}$$
$$\theta = \tan^{-1}\left(\frac{68}{143}\right)$$



Example 3

Particles of weights 5, 7, 6, 8, 4 and 9 N are placed at the angular points of a regular hexagon taken in order. Show that the centre of gravity coincides with the centre of hexagon.



Let the length of each side of the hexagon is $2a$ and O is the centre of the hexagon taken as origin. Also take OC as x - axis and OM as y-axis.

$AB = 2a \Rightarrow OC = 2a = OD$

and $OM = \sqrt{4a^2 - a^2} = \sqrt{3}a$

Let coordinates of centre of gravity be $(\bar{x}, \bar{y})$

Taking moment about OC

$$6.2a + 8.a + 7.a + 4.(-a) + 5.(-a) + 9.(-2a) = (6+8+7+4+5+9)\bar{x}$$
$$\bar{x} = \frac{27a - 27a}{39}$$
$$= 0$$

Taking moment about OM

$$8.a\sqrt{3}+4.a\sqrt{3}+6.0+9.0+5.(-a\sqrt{3})+7.(-a\sqrt{3})=(6+8+7+4+5+9)\overline{y}$$

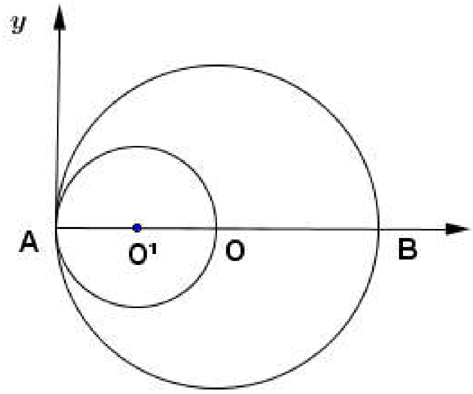
$$\overline{y}=\frac{12a\sqrt{3}-12a\sqrt{3}}{39}$$

$$=0$$

The centre of gravity coincides with the point O which is the centre of the hexagon.

Example 4

A uniform circular disc of radius $\frac{r}{2}$ is cut off from a circle of a radius r of the disc as diameter. Find the centre of gravity of the remainder.



Let AB be the diameter of the circular disc and O its centre.

Let O′ be the centre of the disc described on AO as diameter and w be the weight per unit area.

Weight of the large circulardisc $= \pi r^2 w$

Weight of the small circulardisc $= \pi\left(\frac{r}{2}\right)^2 w=\frac{1}{4}\pi r^2 w$

Let G be the centre of gravity of the remainder.

By symmetry centre of gravity of the remainder lies on AB.

Taking moment about AY,

$$r\left(\pi r^2 w\text{-}\frac{\pi}{4}r^2 w\right)\mathrm{AG}=\pi r^2 w\times \mathrm{AO}\text{ - }\frac{\pi}{4}r^2 w\times \mathrm{AO}^{|}$$

$$=\frac{\pi r^2 w.\ \mathrm{r}\ \text{ - }\ \frac{\pi}{4}r^2 w.\frac{\mathrm{r}}{2}}{\frac{3}{4}\pi r^2 w}$$

$$=\frac{\frac{7}{8}r}{\frac{3}{4}}$$

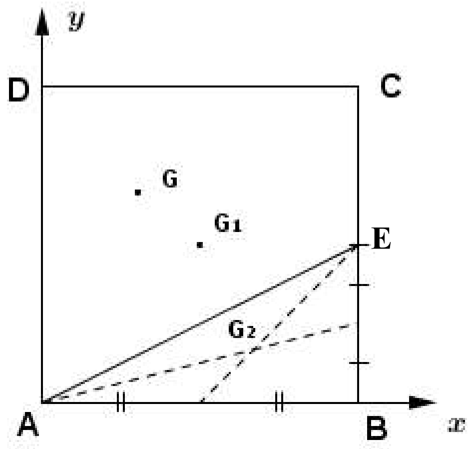
$$=\frac{7}{6}r$$

$$\mathrm{OG}=\frac{7}{6}r-r=\frac{r}{6}$$

Therefore, the distance of the centre of gravity of the remainder from the centre of the original disc is $\frac{1}{6}r$ along the diameter.

Example 5

ABCD is a square lamina of side $2a$. E is the midpoint of the side BC. Find the distance of the centre of gravity of the portion AECD from A.



Let AB and AD are the x, y axes respectively and w be the weight of unit area.

Weight of lamina ABCD is $4a^2w$

Weight of portion ABE is $\frac{1}{2}.2a^2w = a^2w$

Let G_1,G_2 be the centre of gravity of ABCD and ABE respectively and G be the centre of gravity of the portion ABCD.

Let $G=(\bar{x},\bar{y})$

Taking moment about AD,

$$(4a^2w - a^2w)\bar{x} = 4a^2w \times a - a^2w \times \frac{2}{3}\times 2a$$

$$3a^2w\,\bar{x} = \frac{8}{3}a^3w$$

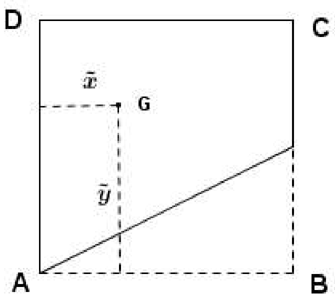
$$\bar{x} = \frac{8}{9}a$$

Taking moment about AB,

$$(4a^2w - a^2w)\bar{y} = 4a^2w \times a - a^2w \times \frac{1}{3}\times a$$

$$3a^2w\,\bar{y} = \frac{11}{3}a^3w$$

$$\bar{y} = \frac{11}{9}a$$



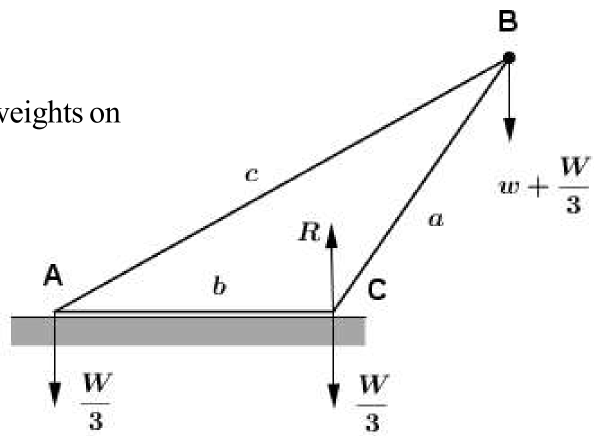
$$AG^2 = \bar{x}^2 + \bar{y}^2$$

$$= \left(\frac{8a}{9}\right)^2 + \left(\frac{11a}{9}\right)^2$$

$$= \frac{\sqrt{185}}{9}a$$

Example 6

A uniform triangular lamina ABC, obtuse angled at C stands in a vertical plane with the side AC in contact with a horizontal table. Show that the largest weight, which if suspended from vertex B will not overturn the lamina is $\frac{1}{3}W\left(\frac{a^2+3b^2-c^2}{c^2-a^2-b^2}\right)$, where W is the weight of the triangle and a, b, c have their usual meanings.



Centre of gravity of the lamina is same as the equal weights on the vertices of the triangle.

So the weight on points A, B, C is $\frac{1}{3}W$

Let w be the largest weight suspended from B. At this stage reaction given by the table to the lamina acts through the point C.

The weight W of the lamina can be considered as three particles of weight $\frac{W}{3}$ placed on vertices A, B and C. As the weight w increases the lamina tends to topple about point C. When w is maximum, reaction R will act at C.

For the equilibrium of the lamina taking moment about C,

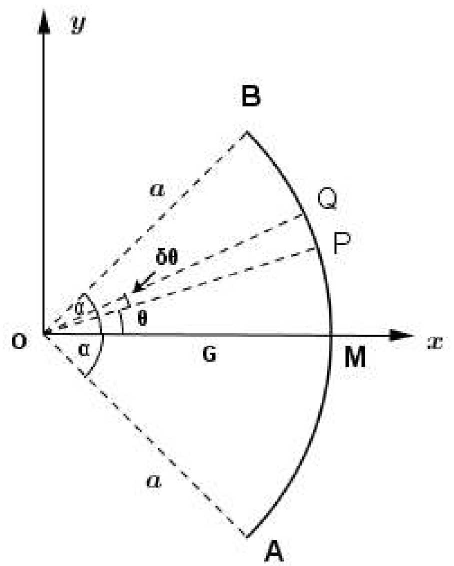
$$\left(w+\frac{W}{3}\right)a\cos(\pi-c)-\frac{W}{3}.b=0$$

$$\left(\frac{w+\frac{W}{3}}{\frac{W}{3}}\right)=\frac{b}{-a\cos c}=\frac{b}{-a\left(\frac{a^2+b^2-c^2}{2ab}\right)}=\frac{2b^2}{c^2-a^2-b^2}$$

$$\frac{w}{\frac{W}{3}}=\frac{2b^2-(c^2-a^2-b^2)}{c^2-a^2-b^2}$$

$$w=\frac{W}{3}\left(\frac{3b^2+a^2-c^2}{c^2-a^2-b^2}\right)$$

Centre of gravity of a circular arc



Let AB be a circular arc and O be the centre of the circle whose radius is a. AB subtends angle 2α at the centre O.

P, Q are the two close points on the arc such that $P\hat{O}Q=\delta\theta$ and $M\hat{O}P=\theta$

M be the midpoint of the arc, $M\hat{O}P=\theta$, w weight per unit length

Weight of elemet $PQ = a\ \delta\theta$ w

Weight of the arc AB = $\int_{-\infty}^{+\infty} a\ d\theta$ w

Centre of gravity of element PQ from O is $a\cos\theta$.

By symmetry centre of the arc AB lies on OM.

Let G be the centre of gravity of the arc AB.

Taking moments about O

$$\left[\int_{-\alpha}^{+\alpha} a\ d\theta\ w\right]OG=\int_{-\alpha}^{+\alpha} a\ d\theta\ w\ .\ a\cos\theta$$

$$aw\int_{-\alpha}^{+\alpha} d\theta\ .OG = a^2w\int\cos\theta\ d\theta$$

$$aw[\theta]_{-\alpha}^{+\alpha}.OG=a^2w[\sin\theta]_{-\alpha}^{+\alpha}$$

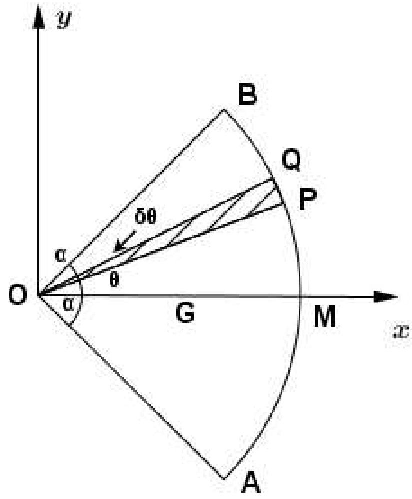
$$aw\ .\ 2\alpha.\ OG=a^2w\ .\ 2\sin\alpha$$

$$OG=\frac{a\sin\alpha}{\alpha}$$

Deduction :

Centre of gravity of semicircular arc, when $\alpha=\frac{\pi}{2}$, $OG=\frac{a\sin\frac{\pi}{2}}{\frac{\pi}{2}}=\frac{2a}{\pi}$

Center of gravity of a sector of a circle



Let AOB be a sector of a circle of radius a and centre O.

Arc AB subtends 2α at O. M be the midpoint of AB.

P, Q are two close point on the arc AB such that $\mathrm{M\hat{O}P} = \theta$ and $\mathrm{P\hat{O}Q} = \delta\theta$. Let m be the weight per unit area

$$\text{weight of } \Delta \text{ POQ} = \frac{1}{2}a^2\delta\theta\ m$$

$$\text{weight of sector AOB} = \int_{-\alpha}^{+\alpha}\frac{1}{2}a^2 d\theta\ m$$

Centre of distance of AOB from O is $\frac{2}{3}a\cos\theta$

By symmetry centre of gravity of the sector G lies on OM

Taking moments about O,

$$\left[\int_{-\alpha}^{+\alpha}\frac{1}{2}a^2 d\theta .m\right]\text{OG} = \int_{-\alpha}^{+\alpha}\frac{1}{2}a^2 d\theta .m.\frac{2}{3}a\cos\theta$$

$$\frac{ma^2}{2}[\theta]_{-\alpha}^{+\alpha}.\text{OG} = \frac{ma^3}{3}[\sin\theta]_{-\alpha}^{+\alpha}$$

$$\frac{ma^2}{2}[2\alpha].\text{OG} = \frac{ma^3}{3}.2\sin\alpha$$

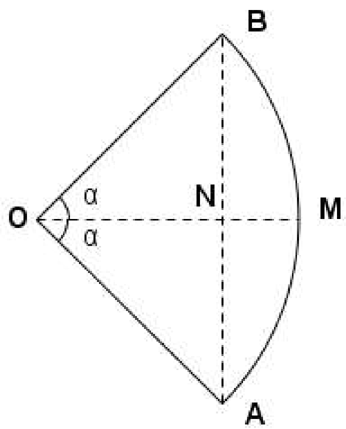
$$\text{OG} = \frac{2}{3}.a\frac{\sin\alpha}{\alpha}$$

Deduction :

Centre of gravity of a semicircular disc.

$$\text{when } \alpha = \frac{\pi}{2},\ \text{OG} = \frac{2}{3}\frac{a\sin\frac{\pi}{2}}{\frac{\pi}{2}} = \frac{4a}{3\pi}$$

Centre of gravity of a segment of a circle



Let AMB is a segment of a circle with centre O and radius a.

By symmetry centre of gravity of the segment G lies on OM.

w - weight of a unit area

Figure	**Weight**	**Centre of gravity from O**
Sector OAMB	$\frac{1}{2}a^2.2\alpha.w$	$\frac{2}{3}a\frac{\sin\alpha}{\alpha}$
Triangle OAB	$\frac{1}{2}.2a\sin\alpha\ .\ a\cos\alpha.\ w$	$\frac{2}{3}a\cos\alpha$
Segment AMB	$a^2(\alpha - \sin\alpha\cos\alpha)w$	OG

Taking moment about O,

$$a^2(\alpha - \sin\alpha\cos\alpha)w\,.\,\mathrm{OG} = \frac{1}{2}a^2\,.\,2\alpha\,.\,w\,.\,\frac{2}{3}a\frac{\sin\alpha}{\alpha} - \frac{1}{2}\,.\,2a\sin\alpha\,.\,a\cos\alpha\,.\,w\,.\,\frac{2}{3}a\cos\alpha$$

$$(\alpha - \sin\alpha\cos\alpha)w\,.\,\mathrm{OG} = \frac{2}{3}a\sin\alpha - \frac{2}{3}a\sin\alpha\cos^2\alpha$$

$$= \frac{2}{3}a\sin\alpha(1-\cos^2\alpha)$$

$$= \frac{2}{3}a\sin^2\alpha$$

$$\mathrm{OG} = \frac{2a\sin^3\alpha}{3(\alpha - \sin\alpha\cos\alpha)}$$

Deduction :

When $\alpha = \frac{\pi}{2}$, segment becomes a semicircular lamina, $\mathrm{OG} = \frac{4a}{3\pi}$

Centre of gravity of a solid hemisphere

Let OM be the axis of symmetry and O is the centre and a the radius of the sphere.

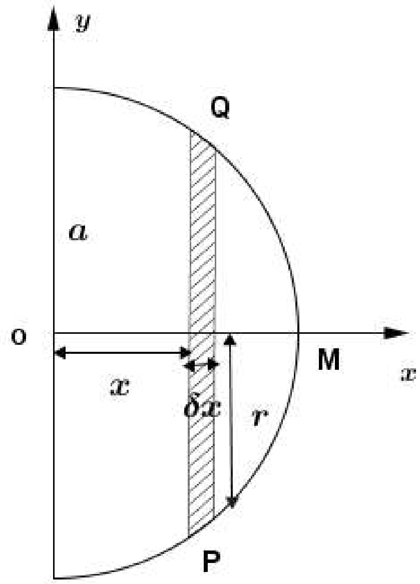
Let PQ be circular disc with thickness δx and in a distance x from O

Let w be the density of the sphere.

Mass of $\mathrm{PQ} = \pi r^2\delta x.\mathrm{w}$

Centre of gravity of PQ, $\pi(a^2 - x^2)\delta x$ w from O is x.

$\therefore$ Mass of the hemisphere $= \int_0^a \pi\left(a^2 - x^2\right)dx$ w



By symmetry, centre of gravity of the hemisphere G lies on OM.

Taking moments about O,

$$\left[\int_0^a \pi\left(a^2-x^2\right)dx.w\right]\mathrm{OG} = \int \pi\left(a^2-x^2\right)dx.w.x$$

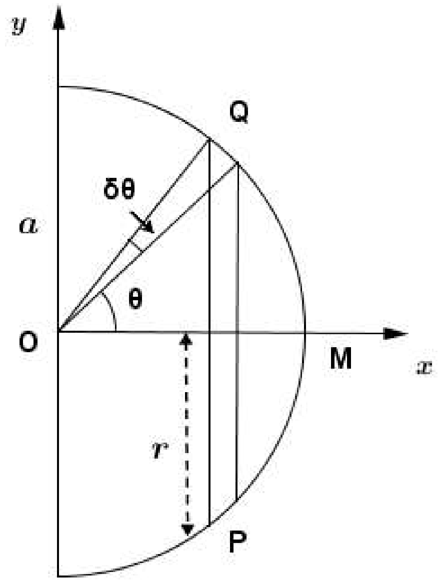
$$\pi w\left[a^2x - \frac{x^3}{3}\right]_0^a \mathrm{OG} = \pi w\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$\pi w \times \frac{2}{3}a^3.\,\mathrm{OG} = \pi w.\,\frac{a^4}{4}$$

$$\mathrm{OG} = \frac{3}{8}a$$

Centre of gravity of a hollow hemisphere

Let OM be the axis of symmetry, O the centre and a the radius of the sphere.



Let PQ be circular ring with the height of a $\delta\theta$ and in a distance $a\cos\theta$ from O.

Also let w be the weight per unit area.

weight of PQ $= (2\pi a\delta\theta)(a\delta\theta).w$

Centre of gravity of PQ from O is $a\cos\theta$.

$\therefore$ Mass of the hollow hemisphere $= \int_0^{\frac{\pi}{2}} 2\pi a \sin\theta \; a \; d\theta \; w$

By symmetry centre of gravity of the hemi sphere G lies on OM

Taking moments about O,

$$\left[\int_0^{\frac{\pi}{2}} 2\pi a \sin\theta \; a \; d\theta \; w\right] \text{OG} = \int_0^{\frac{\pi}{2}} 2\pi a \sin\theta \; a \; d\theta \; wa\cos\theta$$

$$2\pi a^2 w \int_0^{\frac{\pi}{2}} \sin\theta \; d\theta.\ \text{OG} = \pi a^3 w \int_0^{\frac{\pi}{2}} \sin 2\theta \; d\theta$$

$$2\pi a^2 w \left[-\cos\theta\right]_0^{\frac{\pi}{2}} .\text{OG} = \pi a^3 w \left[\frac{-\cos 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

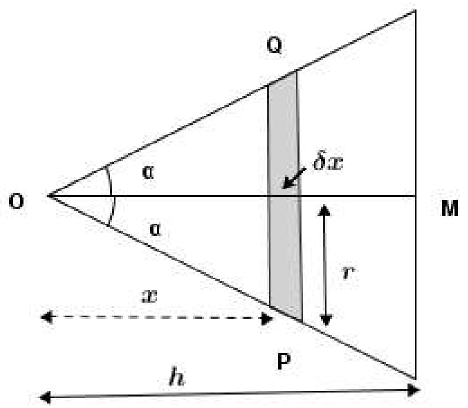
$$2\pi a^2 w\left[0-(-1)\right] .\text{OG} = \pi a^3 w$$

$$\text{OG} = \frac{a}{2}$$

Centre of gravity of a solid cone

Let h be the height and α be semi vertical angle of the cone.

Consider a circular disc PQ with thickness δx and a distance x from vertex O.



Let w be density of the cone

Weight of PQ $= \pi r^2 \delta x$ w g

$= \pi(x\tan\alpha)^2 fx$.w g

Weight of the cone $= \int_0^h \pi x^2 \tan^2\alpha . dx$ w.g

Centre of gravity of PQ from O is x

By symmetry centre of gravity of the cone G lies on OM.

Taking moment about O

$$OG \cdot \left[\int_0^h \pi x^2 \tan^2 \alpha \, dx \, w \right] = \int_0^h \pi x^2 \tan^2 \alpha \, dx \, w.x$$

$$OG \cdot \left[\pi \tan^2 \alpha \; w \int_0^h x^2 dx \right] = \pi \tan^2 \alpha \; w \int_0^h x^3 dx$$

$$OG \cdot \pi \tan^2 \alpha \; w \left| \frac{x^3}{3} \right|_0^h = \pi \tan^2 \alpha \; w \left| \frac{x^4}{4} \right|_0^h$$

$$OG \cdot \frac{\pi}{3} h^3 \tan^2 \alpha \; w = \frac{\pi}{4} h^4 \tan^2 \alpha \; w$$

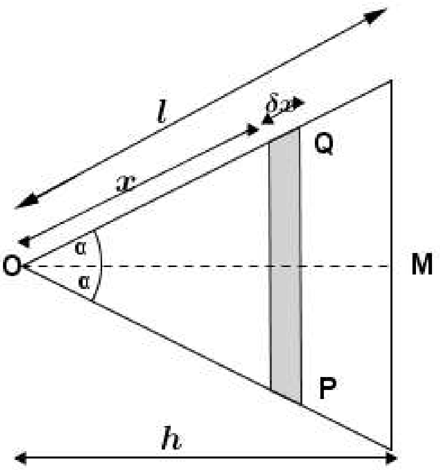
$$\therefore \; OG = \frac{3}{4} h$$

Centre of gravity of a hollow cone

Let h be the height and α be semi-vertical angle of the cone.

Consider a circular ring PQ with a height δx at a distance $x \cos \alpha$

Let w be the weight per unit area.



$$\text{Weight of PQ} \quad = 2\pi \left(x \sin \alpha \right) \delta x.\text{w}$$

$$\text{Weight of the Cone} \quad = \int_0^{\ell} 2\pi \left(x \sin \alpha \right) dx.\text{w}$$

By symmetry centre of gravity of the cone G lies on OM

Taking moments about O,

$$OG . \left[\int_0^{\ell} 2\pi (x \sin \alpha) \, dx.w \right] = \int_0^{\ell} 2\pi x \sin \alpha \, dx \, . \, x \cos \alpha .w$$

$$OG.2\pi \, \sin\alpha .w \int_0^{\ell} x \, dx = 2\pi \sin \alpha \, \cos \alpha \int x^2 dx.w$$

$$OG.2\pi \sin\alpha .w \left| \frac{x^2}{2} \right|_0^{\ell} = 2\pi \sin\alpha \cos \alpha .w \left| \frac{x^3}{3} \right|_0^{\ell}$$

$$OG. \left[2\pi \sin\alpha \; w \; \frac{\ell^2}{2} \right] = 2\pi \sin \alpha \cos \alpha .w \frac{\ell^3}{3}$$

$$OG = \frac{2}{3} \ell \cos \alpha$$

$$OG = \frac{2}{3} h$$

Example 7

From a uniform solid right circular cylinder of radius r and height h, a solid right circular cone of radius r and height $\frac{h}{2}$ is bored out so that the base of the cone coincides with one end of the cylinder. Show that the centre of gravity of the remainder is on the axis at a distance $\frac{23h}{40}$ from the base of the cone.

By symmetry, centre of gravity of the remainder lies on the axis through O.

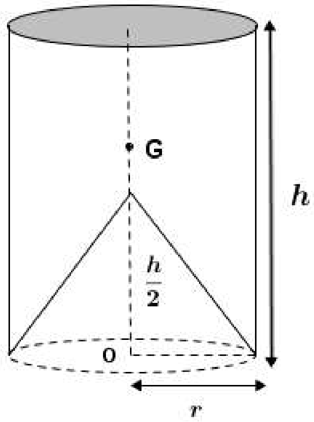


Figure	Weight	Centre of Gravity from O
Cylinder	$\pi r^2 h\rho g$	$\frac{h}{2}$
Cone	$\frac{1}{3}\pi r^2 \frac{h}{2}\rho g$	$\frac{1}{4}\left(\frac{h}{2}\right)=\frac{h}{8}$
Remainder	$\frac{5}{6}\pi r^2 h\rho g$	OG

Taking moment about O,

$$\frac{5}{6}\pi r^2 h\rho g.\text{OG} = \pi r^2 h\rho g\left(\frac{h}{2}\right) - \frac{1}{3}\pi r^2\left(\frac{h}{2}\right)\rho g.\left(\frac{h}{8}\right)$$

$$\frac{5}{6}.\text{OG} = \frac{h}{2}-\frac{h}{48}=\frac{23h}{48}$$

$$\text{OG} = \frac{23h}{40}$$

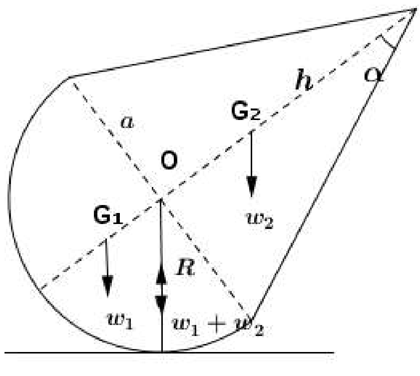
Example 8

A uniform solid body formed by welding together at coincidental bases of radii a, a hemisphere and a right circular cone of semi-vertical angle α . If the body can rest in equlibrium with any point of the curved surface of the hemisphere in contact with a horizontal table, find the value of α .

The body rest in equlibrium with any point of the curved surface of the hemisphere in contact.

Then the reaction through the point of contact and the weight of the whole body $(w_1 + w_2)$ should act through the point of contact and the centre O.

Therefore, taking moment about O



$$\text{w}_1.\ \text{OG}_1 - \text{w}_2.\text{OG}_2 = 0$$

$$\frac{2}{3}\pi a^3\rho.\frac{3}{8}a - \frac{1}{3}\pi a^2 h\rho.\frac{1}{4}h = 0$$

$$3a^2 = h^2$$

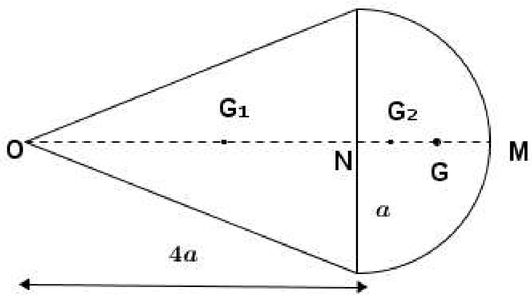
$$\frac{a}{h} = \frac{1}{\sqrt{3}}$$

$$\tan\alpha = \frac{1}{\sqrt{3}}$$

$$\alpha = \frac{\pi}{6}$$

Example 9

A toy is in the form of a composite body formed by joining together a uniform solid right circular cone of density ρ, base radius a and height $4a$ and a uniform solid hemisphere of density $\lambda\rho$ and base radius a so that their bases coincide. Find the distance of the centre of gravity of the toy from the centre of the common base. If the toy cannot be in stable equilibrium with the curved surface of the cone in contact with a smooth horizontal plane, show that $\lambda > 20$.



By symmetry centre of gravity of the toy G lies on OM.

Figure	**Weight**	**Centre of gravity from N**
Cone	$\frac{1}{3}\pi a^2.4a.\rho g$	$NG_1 = -\frac{1}{4}.4a = -a$
Hemisphere	$\frac{2}{3}\pi a^3.\lambda\rho g$	$NG_2 = \frac{3a}{8}$
Toy	$\frac{2}{3}\pi a^3\rho(2+\lambda)g$	NG

Taking moment about O

$$\frac{2}{3}\pi a^3\rho(2+\lambda)g \,.\, NG = \frac{4}{3}\pi a^3\rho g(-a) + \frac{2}{3}\pi a^3\lambda\rho\, g.\frac{3a}{8}$$

$$(2+\lambda).NG = -2a + \frac{3a}{8}\lambda$$

$$NG = \frac{(3\lambda - 16)}{8(2+\lambda)}a$$

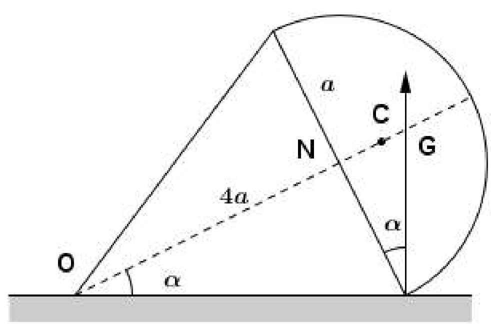
For non stability NC < NG

$$a\tan\alpha < \frac{(3\lambda-16)}{8(2+\lambda)}a$$

$$\frac{1}{4} < \frac{(3\lambda-16)}{8(2+\lambda)}$$

$$2(2+\lambda) < 3\lambda - 16$$

$$20 < \lambda$$

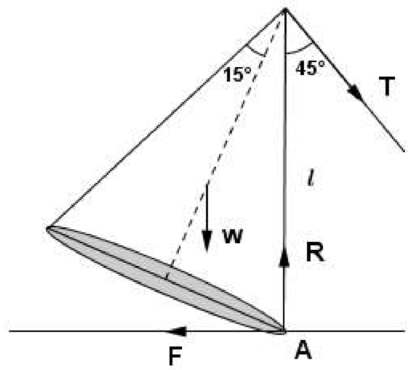


Example 10

A uniform solid cone of semi - vertical angle 15° rests with its base on a rough horizontal floor. It is tilted to one side by a light inextensible string attached to its vertex. The string pulls the cone downwards making an angle 45° with the horizontal, in a vertical plane through the axis of the cone. The edge of the cone is about to slip on the floor, when the vertex is vertically above the point of contact of the edge and the floor. Write down the sufficient equations to determine the tension T in the string, the normal reaction and the frictional force. Hence show that

i. $T = \dfrac{3\sqrt{2}}{16}\mathrm{W}$

ii. The value of the coefficient of friction is $\dfrac{3}{19}$



For the equilibrium of the cone

Taking moment about A

$$\text{A}\curvearrowright \qquad \mathrm{T}.\ell \sin 45° - W.\frac{3}{4}h\sin 15° = 0$$

$$\mathrm{T}.h\sec 15°.\sin 45° - W.\frac{3}{4}h\sin 15° = 0$$

$$\frac{\mathrm{T}}{\sqrt{2}\cos 15°} = \frac{3}{4}W\sin 15°$$

$$\mathrm{T} = \frac{3\sqrt{2}}{8}W\sin 30°$$

$$= \frac{3\sqrt{2}}{16}W$$

Resolving vertically,

$$\uparrow \ \mathrm{R}.\ \mathrm{T}\cos 45° - \mathrm{W} = 0$$

$$\mathrm{R} = \frac{19}{16}W$$

Resolving horizontally,

$$\leftarrow \qquad \mathrm{F} - \mathrm{T}\sin 45° = 0$$

$$\mathrm{F} = \frac{3}{16}W$$

For limiting equilibrium,

$$\frac{\mathrm{F}}{\mathrm{R}} = \mu$$

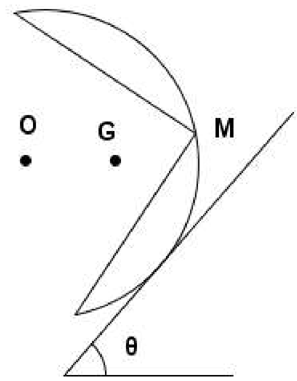
$$\mu = \frac{\frac{3}{16}W}{\frac{19}{16}W}$$

$$= \frac{3}{19}$$

Example 11

A solid is formed by removing a right circular cone of base radius *a* and height a from a uniform solid hemisphere of radius a. The plane base of the hemisphere and the cone are coincidental with O as the common centre of both. Find the distance from O of the centre of the mass G of the solid.

The figure show the cross section of the above solid resting in equilibrium with a point on the curved surface in contact with a rough plane inclined at angle θ to the horizontal. O and G are in same vertical plane through a line of greatest slope of the plane. Given that OG is horizontal. Show that $\theta = 30^0$. Given the weight of the hemisphere is W.



Obtain in terms of W the values of the frictional force and the normal reaction at the point of contact.

Find also the smallest value of the coefficient of friction between the plane and the solid.

By symmetry centre of gravity of the remainder a lies on the axis of the cone.

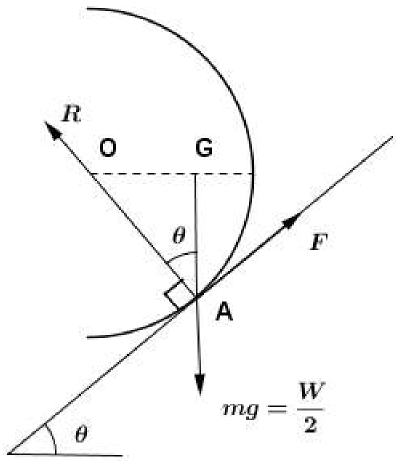
Figure	**Weight**	**Centre of Gravity from O**
Hemisphere	$\frac{2}{3}\pi a^3 \mathrm{W}$	$\frac{3}{8}a$
Cone	$\frac{1}{3}\pi a^2 .a\mathrm{W}$	$\frac{1}{4}a$
Remainder	$\frac{1}{3}\pi a^3 \mathrm{W}$	OG

Taking moment about O,

$$\frac{1}{3}\pi a^3 W.\mathrm{OG} = \frac{2}{3}\pi a^3 W.\frac{3}{8}a - \frac{1}{3}\pi a^3 W.\frac{1}{4}a$$

$$\mathrm{OG} = \frac{6}{8}a - \frac{1}{4}a$$

$$= \frac{a}{2}$$



For the equilibrium of the solid, weight of the solid $\frac{\mathrm{W}}{2}$ should pass

lines of the action of three forces F, R, $\frac{\mathrm{W}}{2}$ through A

$$\therefore \sin\theta = \frac{\mathrm{OG}}{\mathrm{OA}}$$

$$= \frac{\frac{a}{2}}{a}$$

$$= \frac{1}{2}$$

$$\theta = 30°$$

Resolving parallel to the plane,

$$\nearrow \quad F - \frac{W}{2}\sin\theta = 0$$

$$F = \frac{W}{2}\sin\theta$$

$$= \frac{W}{2}\sin 30^\circ$$

$$= \frac{W}{4}$$

Resolving perpendicular to the plane,

$$\nwarrow \quad R - \frac{W}{2}\cos\theta = 0$$

$$R = \frac{W}{2}\cos\theta$$

$$= \frac{W}{2}\cos 30^\circ$$

$$= \frac{W\sqrt{3}}{4}$$

For equilibrium,

$$\frac{F}{R} \le \mu$$

$$\frac{\frac{W}{4}}{\frac{W\sqrt{3}}{4}} \le \mu$$

$$\frac{1}{\sqrt{3}} \le \mu$$

$$\mu_{\min} = \frac{1}{\sqrt{3}}$$

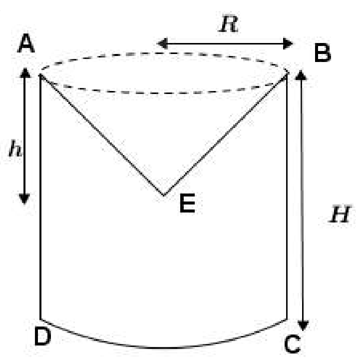
Example 12

The figure shows the remains of a uniform solid right circular cylinder ABCD of height H and base radius R, after solid right circular cone EAB of height h and base radius R is scooped out. Find the distance of gravity of the resulting body S from AB. Hence show that if the centre of gravity of S is at E then $h = \left(2 - \sqrt{2}\right)\mathrm{H}$.

The body S is placed on a rough plane making an angle $\alpha\left(< \frac{\pi}{2}\right)$ with the horizontal, the base DC being on the plane. The plane is rough enough to prevent S from skipping. Assuming that centre of gravity of S is at E show that S will not topple if $\mathrm{R}\cot\alpha > \left(\sqrt{2} - 1\right)\mathrm{H}$.

By symmetry centre of gravity of S lies on the axis of the cylinder.

Figure	**Weight**	**Centre of gravity from**
Cylinder	$\pi \mathrm{R}^2 \mathrm{HW}$	$\frac{\mathrm{H}}{2}$
Cone	$\frac{1}{3}\pi \mathrm{R}^2 h \mathrm{W}$	$\frac{h}{4}$
Body S	$\pi \mathrm{R}^2\left(\mathrm{H} - \frac{h}{3}\right)\mathrm{W}$	$\bar{y}$

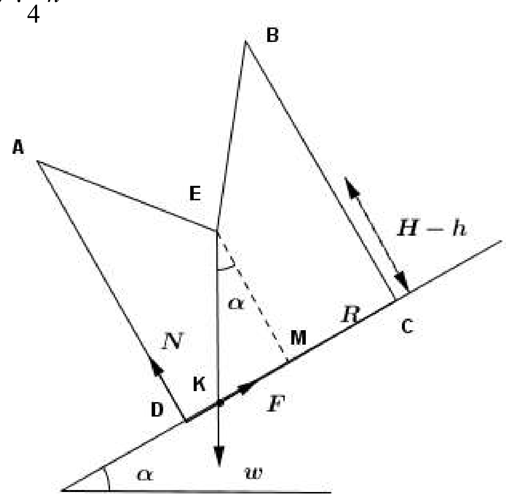


Taking moment about AB

$$\pi R^2\left(H - \frac{h}{3}\right)W\ \overline{y} = \pi R^2 H\ W.\ \frac{H}{2} - \frac{1}{3}\pi R^2 h W.\frac{1}{4}h$$

$$\left(H - \frac{h}{3}\right)\ \overline{y} = \frac{H^2}{2} - \frac{h^2}{12}$$

$$\overline{y} = \frac{6H^2 - h^2}{4(3H - h)}$$



If centre of gravity is on E, then $\overline{y} = h$

$$h = \frac{6H^2 - h^2}{4(3H-h)}$$

$$\Rightarrow 3h^2 - 12Hh + 6H^2 = 0$$

$$h^2 - 4Hh + 2H^2 = 0$$

$$(h - 2H)^2 - 2H^2 = 0$$

$$\left(h - 2H+\sqrt{2}H\right)\left(h - 2H - \sqrt{2}H\right) = 0$$

$$h = 2H - \sqrt{2}H,\quad 2H+\sqrt{2}H$$

$$\Rightarrow h < H \Rightarrow h = 2H - \sqrt{2}H$$

$$= \left(2 - \sqrt{2}\right)H$$

If KM < DM, the body will not topple.

$$(H - h)\tan\alpha < R$$

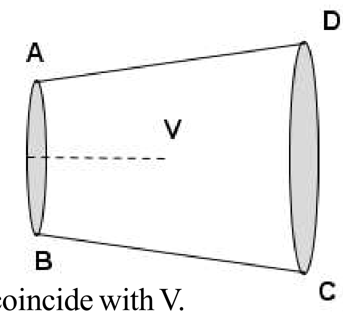
$$(H - h) < R\cot\alpha$$

$$\left(\sqrt{2} - 1\right)H < R\cot\alpha$$

Example 13

In the figure below, ABCD represents a uniform solid body of density ρ in the form of a frustum of height h of a right circular cone. The diameters of its circular plane faces are $AB = 2a\lambda$, and $CD = 2a$ where λ is a parameter and $0 < \lambda < 1$.

Show by intergration that its mass is $\frac{1}{3}\pi a^2 h\rho(1+\lambda+\lambda^2)$ and that its centre of mass G is at a distance $\frac{h}{4}\left(\frac{3+2\lambda+\lambda^2}{1+\lambda+\lambda^2}\right)$ from the centre of the smaller face.



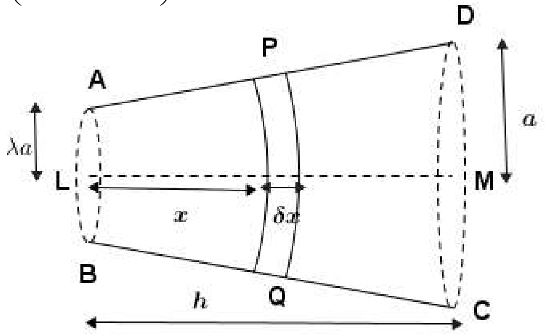
Deduce the mass and the position of the centre of mass of uniform right circular solid cone of base radius a and height h.

A solid body J is obtained frustum ABCD by scooping out a right circular solid cone VAB of base radius λa and hight $\frac{h}{2}$.

Find the position of the centre of mass G_1 of J and verify that G_1 cannot coincide with V.

The body J is suspended freely from a point on the circumference of the larger face. Show that in the position of equilibrium the axis of symmetry of J makes and acute angle β with the vertical given by

$$\tan\beta = \frac{8a}{h}\left(\frac{2+2\lambda+\lambda^2}{4+8\lambda+5\lambda^2}\right)$$



$$\frac{x}{r-\lambda a} = \frac{h}{a(1-\lambda)}$$

$$r-\lambda a = \frac{a(1-\lambda)}{h}x$$

$$r = \frac{a(1-\lambda)}{h}x + \lambda a$$

Consider circular disc PQ with height of δx at a distance x from AB.

Volume of PQ $= \pi r^2 \delta x$

Mass of PQ $= \pi r^2 \delta x \rho$

Mass of the frustum $= \int_0^h \pi r^2 dx \rho$

$$\therefore \quad = \int_0^h \pi.\left[\frac{a(1-\lambda)x}{h} + \lambda a\right]^2 dx\rho = \pi\rho\left|\frac{\left[\frac{a(1-\lambda)x}{h}+\lambda a\right]^3}{3a\frac{(1-\lambda)}{h}}\right|_0^h$$

$$= \frac{\pi\rho}{3}\frac{h}{a(1-\lambda)}\left\{\left[a(1-\lambda)+\lambda a\right]^3 - (\lambda a)^3\right\}$$

$$= \frac{\pi\rho}{3}\frac{ha^3(1-\lambda^3)}{a(1-\lambda)} = \frac{\pi}{3}a^2h\rho\frac{(1-\lambda^3)}{(1-\lambda)}$$

$$M = \frac{\pi}{3}a^2h(1+\lambda+\lambda^2)\rho \text{..........................(1)}$$

By symmetry centre of mass G lies on the line connecting the centre of bases.

$$M.\ LG = \int \pi r^2 \delta x \rho . x$$

$$LG = \frac{\int_0^h \pi\rho\left[\frac{a(1-\lambda)}{h}x + \lambda a\right]^2 x dx}{M}$$

$$= \frac{\pi\rho}{M}\int_0^h \left[\frac{a^2(1-\lambda)^2}{h^2}x^3 + \frac{2\lambda a^2}{h}(1-\lambda)x^2 + \lambda^2 a^2 x\right]dx$$

$$= \frac{\pi\rho}{M}\left[\frac{a^2(1-\lambda)^2}{h^2}\frac{x^4}{4} + 2\lambda\frac{a^2}{h}(1-\lambda)\left(\frac{x^3}{3}\right) + \lambda^2 a^2\frac{x^2}{2}\right]_0^h$$

$$= \frac{\pi\rho}{M}\left[\frac{a^2(1-\lambda)^2}{h^2}\frac{x^4}{4} + 2\lambda\frac{a^2}{h}(1-\lambda)\left(\frac{x^3}{3}\right) + \lambda^2 a^2\frac{x^2}{2}\right]$$

$$= \frac{\pi\rho}{M}h^2a^2\left[\frac{(1-\lambda)^2}{4} + \frac{2}{3}\lambda(1-\lambda) + \frac{\lambda^2}{2}\right]$$

$$= \frac{\pi a^2 h^2 \rho}{M}\left[\frac{3(1-2\lambda+\lambda^2)+8\lambda+8\lambda^2+6\lambda^2}{4\times 3}\right]$$

$$=\frac{\pi a^2 h^2 \rho}{12M}\left(\lambda^2+2\lambda+3\right)$$

$$=\frac{\pi a^2 h^2 \rho}{12}\frac{\left(\lambda^2+2\lambda+3\right)}{\frac{\pi}{3}a^2 h\left(1+\lambda+\lambda^2\right)\rho}$$

$$=\frac{h}{4}\left(\frac{\lambda^2+2\lambda+3}{\lambda^2+\lambda+1}\right) \quad \text{..........................(2)}$$

When $\lambda=0$ the frustum becomes a cone with height of h and base radius a.

$\therefore$ Mass of the cone $=\frac{1}{3}\pi a^2 h\rho$ from (1) with $\lambda=0$

Centre of the mass of the cone from the vertex is $\frac{h}{4}.\frac{3}{1}=\frac{3h}{4}$ [from (2) of $\lambda=0$]

Finding the centre of gravity of J

Figure	**Weight**	**Centre of gravity from AB**
Frustum	$\frac{1}{3}\pi a^2\rho g(1+\lambda+\lambda^2)h$	$\frac{h}{4}\left(\frac{\lambda^2+2\lambda+3}{\lambda^2+\lambda+1}\right)$
Cone VAB	$\frac{1}{3}\pi\left(\lambda a\right)^2\rho g\frac{h}{2}$	$\frac{1}{4}.\frac{h}{2}=\frac{h}{8}$
Remainder	$\frac{1}{3}\pi a^2 h\rho g\left(1+\lambda+\frac{\lambda^2}{2}\right)$	$\bar{y}$

Taking moment about L

$$\frac{1}{3}\pi a^2 h\rho g\left(1+\lambda+\frac{\lambda^2}{2}\right)\bar{y}=\frac{1}{3}\pi a^2\rho g\left(1+\lambda+\lambda^2\right).\frac{h}{4}\left(\frac{\left(\lambda^2+2\lambda+3\right)}{\lambda^2+\lambda+1}\right)-\frac{1}{3}\pi a^2\lambda^2\rho\frac{h}{2}g.\frac{h}{8}$$

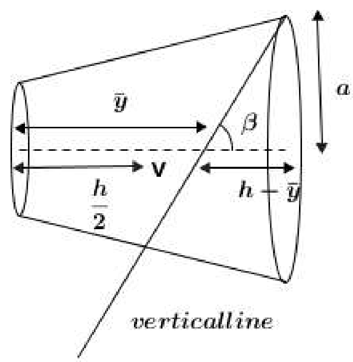
$$\left(1+\lambda+\frac{\lambda^2}{2}\right)\bar{y}=\frac{h}{4}\left(\lambda^2+2\lambda+3\right)-\frac{h\lambda^2}{16}$$

$$\bar{y}=\frac{h}{8}\left(\frac{3\lambda^2+8\lambda+12}{\lambda^2+2\lambda+2}\right)$$

$$\bar{y}-\frac{h}{2}=\frac{h}{8}\left(\frac{3\lambda^2+8\lambda+12}{\lambda^2+2\lambda+2}\right)-\frac{h}{2}$$

$$=\frac{h}{8}\left[\frac{3\lambda^2+8\lambda+12-4\left(\lambda^2+2\lambda+2\right)}{\lambda^2+2\lambda+2}\right]$$

$$=\frac{h}{8}\left(\frac{4-\lambda^2}{\lambda^2+2\lambda+2}\right)>0 \qquad (\because 0<\lambda<1)$$



$\therefore$ Point V cannot coincide with G_1

$$\tan\beta=\frac{a}{h-\bar{y}}$$

$$h-\bar{y}=h-\frac{h}{8}\left(\frac{3\lambda^2+8\lambda+12}{2+2\lambda+\lambda^2}\right)$$

$$=\frac{h}{8}\left(\frac{5\lambda^2+8\lambda+4}{2+2\lambda+\lambda^2}\right)$$

$$\therefore\ \tan\beta=\frac{8a}{h}\left(\frac{2+2\lambda+\lambda^2}{4+8\lambda+5\lambda^2}\right)$$

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8.4 Exercises

1. From a uniform triangle ABC, a portion ADE is removed where DE is parallel to BC and the area of the triangle ADE equals to half of ABC. Find the centre of gravity of the remainder from BC.

2. From a triangle ABC, a portion ADE, where DE is parallel to BC, is removed. If a and b be the distances of A from BC and DE respectively, show that the distance of the centre of gravity of the remainder from BC is $\frac{a^2+ab-2b^2}{3(a+b)}$.

3. Three rods of length a, b, c are joined at their ends so as to form a triangle. Find the centre of gravity of the triangle.

4. From a uniform triangular board ABC a portion consisting of the area of the inscribed circular is removed. Show that the distance of the centre of gravity of the remainder from BC is $\frac{S}{3as}\left[\frac{2s^3-3\pi aS}{s^2-\pi S}\right]$ where S is the area, s the semi-perimeter of the board and BC = a.

5. ACB is a uniform semicircular lamina with diameter AOB, and OC is the radius perpendicular to AB. A square portion OPQR is cut off from the lamina, P being on OB and length of OP is $\frac{1}{2}a$. Find the distance from OA and OC of the centre of gravity of the remaining portion. Hence show that if the remaining portion is suspended from A and hangs in equilibrium, the tangent of the angle made by AB with the vertical is just less than $\frac{1}{2}$.

6. ABCDEF is a sheet of thin cardboard in the form of a regular hexagon. Prove that if the triangle ABC is cut off and superposed on the triangle DEF, the centre of gravity of the whole is moved by a distance $\frac{2a}{9}$, where a is the side of the hexagon.

7. Prove that the centre of gravity of a uniform semicircular lamina of radius a is at a distance $\frac{4a}{3\pi}$ from its centre.
   AOB is the base of a uniform semicircular lamina of radius $2a$, O being its centre. A semicircular lamina of radius a and base AO is cut away and the remainder is suspended freely from A. Find the inclination of AOB to the vertical in the equilibrium position.

8. A solid cylinder and a solid right circular cone have their bases joined together, the bases being of the same size. Find the ratio of the height of the cone to the height of the cylinder so that the centre of gravity of the compound solid may be at the centre of the common base.

9. A solid in the form of a right circular cone has its base scooped out, so that the hollow formed is a right circular cone on the same base. How much must be scooped out so that the centre of gravity of the remainder may coincide with the vertex of the hollow cone.

10. From a uniform solid right circular cone of vertical angle 60° is cut out the greatest possible sphere. Show that centre of gravity of the remainder divides the axis in the ratio 11 : 49.

11. A solid right circular cone of height h is cut off at a height $\frac{1}{2}h$ by a plane perpendicular to the axis. Find the centre of the gravity of the portion between this section and the base of the cone.

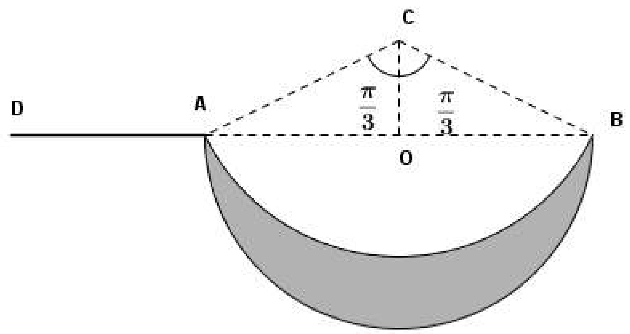
12. A hollow vessel made of uniform material of negligible thickness is in the form of a right circular cone of surface density ρ mounted on a hemisphere of surface density σ whose radius is equal to that of the circular rim of the cone. If the vessel can just rest with a generator of the cone in contact with a smooth horizontal plane, prove that semi-vertical angle a of the cone is given by the equation $\rho\left(\cot^2 \alpha+3\right)=3\sigma\left(\cos\alpha-2\sin\alpha\right)$.

13. A hollow baseless cone of vertex O, semi-vertical angle α and height h is made of a uniform thin metal sheet of mass σ per unit area. Show that its mass is $\pi\sigma h^2 \sec\alpha \tan\alpha$ and find the position of its centre of mass.

    A uniform circular disc of centre B and radius $h\tan\alpha$ made of the same metal sheet is now fixed as the base of the above cone. Show that the distance of the centre of mass of the composite body from O is $\dfrac{h\left(\frac{2}{3}\sec\alpha+\tan\alpha\right)}{\sec\alpha+\tan\alpha}$.

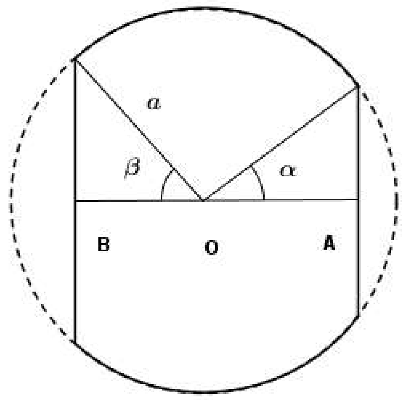
    The composite body is suspended from a point A on the rim of the base. If AO and AB make equal angles with the downward vertical show that $\sin\alpha=\frac{1}{3}$.

14. A crescent shaped uniform lamina is bounded by a semicircle with centre O and radius a and a circular arc subtending an angle $\frac{2\pi}{3}$ at its centre C as shown in the figure. Show that the centre of mass of this lamina is at a distance ka from C, where $k=\dfrac{3\sqrt{3}\pi}{\pi+6\sqrt{3}}$



    Let M be the mass of the lamina. The end A of a thin uniform straight rod AD of length $2a$ and mass m is rigidly fixed to the crescent at A along the extended line BA, forming a sickle as show in the figure. The sickle is then placed on a horizontal floor with the pan of lamina vertical and the semicircle and the free end D of the rod touching the floor. If it stays in equilibrium in this position show that $M\left(\sqrt{3}k-1\right)<4\sqrt{6}\,m$.

15. Out of a uniform spherical shell of radius a, centre O, and surface density σ, a cone is cut off by two parallel planes at distances $a\cos\alpha, a\cos\beta$ from O (on either side of O) where $0<\alpha<\beta<\frac{\pi}{2}$ as shown in the figure.



Show by intergration, that

(i) The mass of the is cone is $2\pi a^2\sigma(\cos\alpha+\cos\beta)$

(ii) The centre of mass of the cone lies on the axis of symmetry midway between its two ends A, B with the end A at a distance $a\cos\alpha$ from O.

A thin uniform circular disc of the same surface density σ and radius $a\sin\beta$ is now fastened to the larger circular edge of the cone so that the centre of the disc is at B. Show that the composite body can rest in equilibrium with any point of the spherical surface on a horizontal floor provided that $\sin\alpha=\sqrt{1-\cos\beta}$.

16. Show by intergration that the centre of the gravity of the frustum obtained by cutting a uniform hollow hemisphereical shell of radius a and surface density σ by a plane parallel to its circular rim and at a distance a cos α from the centre O is at the mid-point of OC where C is the centre of the smaller circular rim.

A bowl is made by rigidly fixing the edge of a thin uniform circular plate of radius a sin a having the same surface density s to the smaller circular rim of the above frustum. Show that the centre of gravity the bowl is on OC at a distance $\left[\dfrac{1+\cos\alpha-\cos^2\alpha}{1+2\cos\alpha-\cos^2\alpha}\right]a\cos\alpha$ from O.

Let $\alpha=\dfrac{\pi}{3}$ and let w be the weight of the bowl. A saucepan is made by rigidly fixing a thin uniform rod AB of length b and weight $\dfrac{w}{4}$ to the rim of the board as a handle such that the points O, A and B are collinear as shown in the figure. Find the position of the centre of gravity of the saucepan..

The saucepan is freely suspended from the end B of the handle and hangs in equilibrium with the handle making an angle $\tan^{-1}\left(\dfrac{1}{7}\right)$ with the downward vertical. Show that $3b=4a$.