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இலங்கைப் பரீட்சைத் திணைக்களம்
Department of Examinations, Sri Lanka

අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, 2021(2022)
கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2021(2022)
General Certificate of Education (Adv. Level) Examination, 2021(2022)

සංයුක්ත ගණිතය I
இணைந்த கணிதம் I
Combined Mathematics I

10 E I

පැය තුනයි
மூன்று மணித்தியாலம்
Three hours

අමතර කියවීම් කාලය - මිනිත්තු 10 යි
மேலதிக வாசிப்பு நேரம் - 10 நிமிடங்கள்
Additional Reading Time - 10 minutes

Use additional reading time to go through the question paper, select the questions you will answer and decide which of them you will prioritise.

Index Number | | | | | | | |

Instructions:

- *This question paper consists of two parts;* **Part A** (Questions 1 - 10) and **Part B** (Questions 11 - 17).
- **Part A:** *Answer **all** questions. Write your answers to each question in the space provided. You may use additional sheets if more space is needed.*
- **Part B:** *Answer **five** questions only. Write your answers on the sheets provided.*
- *At the end of the time allotted, tie the answer scripts of the two parts together so that* **Part A** *is on top of* **Part B** *and hand them over to the supervisor.*
- *You are permitted to remove **only*** **Part B** *of the question paper from the Examination Hall.*

For Examiners' Use only

(10) Combined Mathematics I		
Part	**Question No.**	**Marks**
A	1	
	2	
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B	11	
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	17	
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Total

In Numbers	
In Words	

Code Numbers

Marking Examiner		
Checked by:	1	
	2	
Supervised by:		

[see page two

Part A

1. Using the **Principle of Mathematical Induction**, prove that $\sum_{r=1}^{n}(6r+1) = n(3n+4)$ for all $n \in \mathbb{Z}^+$.

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2. Sketch the graphs of $y = 2|x+1|$ and $y = 2-|x|$ in the same diagram.
Hence or otherwise, find all real values of x satisfying the inequality $2|x+2|+|x| \leq 4$.

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[*see page three*

3. Sketch, in an Argand diagram, the locus of the points that represent complex numbers z satisfying $\operatorname{Arg}(z-1-i) = -\frac{\pi}{4}$.
Hence or otherwise, show that the minimum value of $|z-2+i|$ satisfying $\operatorname{Arg}(iz+1-i) = \frac{\pi}{4}$ is $\frac{1}{\sqrt{2}}$.

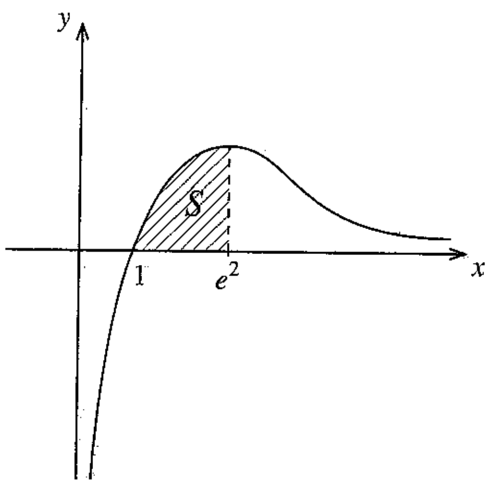
4. Let $k > 0$. It is given that the coefficient of x^7 in the binomial expansion of $\left(x^2 + \frac{k}{x}\right)^{11}$ and the coefficient of x^{-7} in the binomial expansion of $\left(x - \frac{1}{x^2}\right)^{11}$ are equal. Show that $k = 1$.

[see page four

5. Show that $\lim_{x \to 0} \dfrac{\tan 2x - \sin 2x}{x^2\left(\sqrt{1+x} - \sqrt{1-x}\right)} = 4$.

6. Let S be the region enclosed by the curves $y = \dfrac{\ln x}{\sqrt{x}}$, $y = 0$ and $x = e^2$. Show that the area of S is 4 square units.

The region S is rotated about the x–axis through 2π radians. Show that the volume of the solid thus generated is $\dfrac{8\pi}{3}$.



[see page five

7. Show that the equation of the tangent line to the rectangular hyperbola parametrically given by $x = ct$ and $y = \frac{c}{t}$ for $t \neq 0$, at the point $P \equiv \left(cp, \frac{c}{p}\right)$ is given by $x + p^2 y = 2cp$.

The normal line to this hyperbola at P meets the hyperbola again at another point $Q \equiv \left(cq, \frac{c}{q}\right)$.

Show that $p^3 q = -1$.

8. Let $A \equiv (0, -1)$ and $B \equiv (9, 8)$. The point C lies on AB such that $AC:CB = 1:2$. Show that the equation of the straight line l through C perpendicular to AB is $x + y - 5 = 0$.

Let D be the point on l such that AD is parallel to the straight line $y = 5x + 1$. Find the coordinates of D.

[see page six

9. Show that the straight line $x + 2y = 3$ intersects the circle $S \equiv x^2 + y^2 - 4x + 1 = 0$ at two distinct points.

Find the equation of the circle passing through these two points and the centre of the circle $S = 0$.

10. Express $2\cos^2 x + 2\sqrt{3}\sin x \cos x - 1$ in the form $R\cos(2x - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

Hence, solve the equation $\cos^2 x + \sqrt{3}\sin x \cos x = 1$.

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[see page seven

අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, 2021(2022)
கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2021(2022)
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සංයුක්ත ගණිතය I
இணைந்த கணிதம் I
Combined Mathematics I

10 E I

Part B

* Answer **five** questions only.

11. (*a*) Let $k>1$. Show that the equation $x^2-2(k+1)x+(k-3)^2=0$ has real distinct roots.
Let α and β be these roots. Write down $\alpha+\beta$ and $\alpha\beta$ in terms of k, and find the values of k such that both α and β are positive.
Now, let $1<k<3$. Find the quadratic equation whose roots are $\dfrac{1}{\sqrt{\alpha}}$ and $\dfrac{1}{\sqrt{\beta}}$, in terms of k.

(*b*) Let $f(x)=2x^3+ax^2+bx+1$ and $g(x)=x^3+cx^2+ax+1$, where $a, b, c \in \mathbb{R}$. It is given that the remainder when $f(x)$ is divided by $(x-1)$ is 5, and that the remainder when $g(x)$ is divided by x^2+x-2 is $x+1$. Find the values of a, b and c.
Also, with these values for a, b and c, show that $f(x)-2g(x)\le \dfrac{13}{12}$ for all $x\in\mathbb{R}$.

12. (*a*) It is required to form a 4-digit number consisting of 4 digits taken from the 10 digits given below:

1, 1, 1, 2, 2, 3, 3, 4, 5, 5

Find the number of different such 4-digit numbers that can be formed
(i) if all 4 digits chosen are different,
(ii) if any 4 digits can be chosen.

(*b*) Let $U_r=\dfrac{-16r^3+12r^2+40r+9}{5(2r+1)^2(2r-1)^2}$ for $r\in\mathbb{Z}^+$.

Determine the values of the real constants A and B such that $U_r=\dfrac{A(r-1)}{(2r+1)^2}-\dfrac{(r-B)}{(2r-1)^2}$ for $r\in\mathbb{Z}^+$.

Hence, find $f(r)$ such that $\dfrac{1}{5^{r-1}}U_r=f(r)-f(r-1)$ for $r\in\mathbb{Z}^+$, and

show that $\displaystyle\sum_{r=1}^{n}\frac{1}{5^{r-1}}U_r=1+\frac{n-1}{5^n(2n+1)^2}$ for $n\in\mathbb{Z}^+$.

Deduce that the infinite series $\displaystyle\sum_{r=1}^{\infty}\frac{1}{5^{r-1}}U_r$ is convergent and find its sum.

[see page eight

13. (*a*) Let $\mathbf{A} = \begin{pmatrix} a & 0 & 3 \\ 0 & a & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} a & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$, where $a \in \mathbb{R}$.

Also, let $\mathbf{C} = \mathbf{A}\mathbf{B}^{\mathrm{T}}$. Find $\mathbf{C}$ in terms of a, and show that $\mathbf{C}^{-1}$ exists for all $a \neq 0$.

Write down $\mathbf{C}^{-1}$ in terms of a, when it exists.

Show that if $\mathbf{C}^{-1}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 9 \\ -11 \end{pmatrix}$, then $a = 2$.

With this value for a, find the matrix $\mathbf{D}$ such that $\mathbf{DC} - \mathbf{C}^{\mathrm{T}}\mathbf{C} = 8\mathbf{I}$, where $\mathbf{I}$ is the identity matrix of order 2.

(*b*) Let $z_1 = 1 + \sqrt{3}\,i$ and $z_2 = 1 + i$. Express $\frac{z_1}{z_2}$ in the form $x + iy$, where $x, y \in \mathbb{R}$.

Also, express each of the complex numbers z_1 and z_2 in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $0 < \theta < \frac{\pi}{2}$, and **hence**, show that $\frac{z_1}{z_2} = \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$.

Deduce that $\cos\left(\frac{\pi}{12}\right) = \frac{1+\sqrt{3}}{2\sqrt{2}}$.

(*c*) Let $n \in \mathbb{Z}^+$ and $\theta \neq 2k\pi \pm \frac{\pi}{2}$ for $k \in \mathbb{Z}$.

Using De Moivre's theorem, show that $(1 + i\tan\theta)^n = \sec^n\theta(\cos n\theta + i\sin n\theta)$.

Hence, obtain a similar expression for $(1 - i\tan\theta)^n$, and

show that $(1 + i\tan\theta)^n + (1 - i\tan\theta)^n = 2\sec^n\theta\cos n\theta$.

Deduce that $z = i\tan\left(\frac{\pi}{10}\right)$ is a solution of $(1+z)^{25} + (1-z)^{25} = 0$.

14. (*a*) Let $f(x) = \frac{4x+1}{x(x-2)}$ for $x \neq 0, 2$.

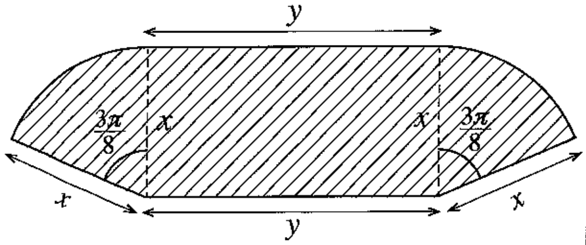
Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x) = -\frac{2(2x-1)(x+1)}{x^2(x-2)^2}$ for $x \neq 0, 2$.

Hence, find the intervals on which $f(x)$ is increasing and the intervals on which $f(x)$ is decreasing.

Sketch the graph of $y = f(x)$ indicating the asymptotes, x-intercept and the turning points.

Using this graph, find all real values of x satisfying the inequality $f(x) + |f(x)| > 0$.

(*b*) The shaded region S of the adjoining figure shows a garden consisting of a rectangle and two sectors of a circle each subtending an angle $\frac{3\pi}{8}$ at the centre. Its dimensions, in metres, are shown in the figure. The area of S is given to be $36\ \mathrm{m}^2$. Show that the perimeter p m of S is given by $p = 2x + \frac{72}{x}$ for $x > 0$ and that p is minimum when $x = 6$.



[see page nine

15. (*a*) Find the values of the constants A, B and C such that

$x^4 + 3x^3 + 4x^2 + 3x + 1 = A(x^2+1)^2 + Bx(x^2+1) + Cx^2$ for all $x \in \mathbb{R}$.

Hence, write down $\dfrac{x^4+3x^3+4x^2+3x+1}{x(x^2+1)^2}$ in partial fractions and

find $\displaystyle\int \frac{x^4+3x^3+4x^2+3x+1}{x(x^2+1)^2}\,\mathrm{d}x$.

(*b*) Let $I = \displaystyle\int_0^{\frac{1}{4}} \sin^{-1}\left(\sqrt{x}\right)\mathrm{d}x$. Show that $I = \dfrac{\pi}{24} - \dfrac{1}{2}\displaystyle\int_0^{\frac{1}{4}} \sqrt{\frac{x}{1-x}}\,\mathrm{d}x$ and **hence**, evaluate I.

(*c*) Show that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x\ln(x^2+1) + 2\tan^{-1}x - 2x\right) = \ln(x^2+1)$.

Hence, find $\displaystyle\int \ln(x^2+1)\,\mathrm{d}x$ and show that $\displaystyle\int_0^1 \ln(x^2+1)\,\mathrm{d}x = \frac{1}{2}(\ln 4 + \pi - 4)$.

Using the result $\displaystyle\int_0^a f(x)\,\mathrm{d}x = \int_0^a f(a-x)\,\mathrm{d}x$, where a is a constant,

find the value of $\displaystyle\int_0^1 \ln\left[(x^2+1)(x^2-2x+2)\right]\mathrm{d}x$.

16. Let $P \equiv (x_1, y_1)$ and l be the straight line given by $ax + by + c = 0$. Show that the coordinates of any point on the line through the point P and perpendicular to l are given by $(x_1 + at, y_1 + bt)$, where $t \in \mathbb{R}$.

Deduce that the perpendicular distance from P to l is $\dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$.

Let l be the straight line $x + y - 2 = 0$. Show that the points $A \equiv (0, 6)$ and $B \equiv (3, -3)$ lie on opposite sides of l.

Find the acute angle between l and the line AB.

Find the equations of the circles S_1 and S_2 with centres at A and B, respectively, and touching l.

Let C be the point of intersection of l and the line AB. Find the coordinates of the point C.

Find also the equation of the other common tangent through C to S_1 and S_2.

Show that the equation of the circle that passes through the origin, bisects the circumference of S_1 and orthogonal to S_2 is $3x^2 + 3y^2 - 38x - 22y = 0$.

[see page ten

17. (*a*) Write down $\cos(A+B)$ and $\cos(A-B)$ in terms of $\cos A$, $\cos B$, $\sin A$ and $\sin B$.

Hence, show that $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

Deduce that $\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$.

Solve the equation $\cos 9x + \cos 7x + \cot x\,(\cos 9x - \cos 7x) = 0$.

(*b*) In the usual notation, state and prove the **Cosine Rule** for a triangle ABC.

Let $x \neq n\pi + \frac{\pi}{2}$ for $n \in \mathbb{Z}$. Show that $\sin 2x = \frac{2\tan x}{1+\tan^2 x}$.

In a triangle ABC, it is given that $AB = 20$ cm, $BC = 10$ cm and $\sin 2B = \frac{24}{25}$.

Show that there are two distinct such triangles and find the length of AC for each.

(*c*) Solve the equation $\sin^{-1}\left[(1+e^{-2x})^{-\frac{1}{2}}\right] + \tan^{-1}(e^{x}) = \tan^{-1}(2)$.

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இலங்கைப் பரீட்சைத் திணைக்களம்
Department of Examinations, Sri Lanka

අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, 2021(2022)
கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2021(2022)
General Certificate of Education (Adv. Level) Examination, 2021(2022)

සංයුක්ත ගණිතය II
இணைந்த கணிதம் II
Combined Mathematics II

10 E II

පැය තුනයි
மூன்று மணித்தியாலம்
Three hours

අමතර කියවීම් කාලය - මිනිත්තු 10 යි
மேலதிக வாசிப்பு நேரம் - 10 நிமிடங்கள்
Additional Reading Time - 10 minutes

Use additional reading time to go through the question paper, select the questions you will answer and decide which of them you will prioritise.

Index Number | | | | | | | |

Instructions:

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* In this question paper, g denotes the acceleration due to gravity.

For Examiners' Use only

(10) Combined Mathematics II		
Part	**Question No.**	**Marks**
A	1	
	2	
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B	11	
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Code Numbers

Marking Examiner	
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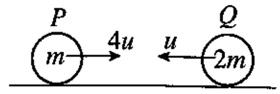
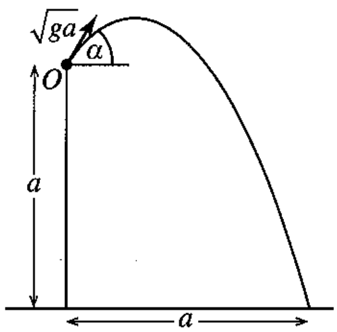
Part A

1. A particle P of mass m and a particle Q of mass $2m$ moving on a smooth horizontal table along the same straight line towards each other with speeds $4u$ and u, respectively, collide directly. The coefficient of restitution between P and Q is $\frac{4}{5}$. Show that the particles P and Q move away from each other after the collision. Find the time taken, after the collision, for P and Q to be at a distance a apart.

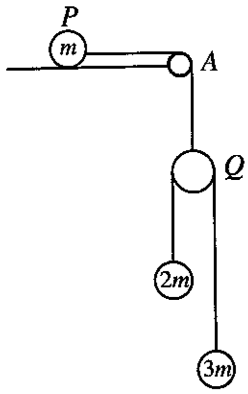


2. A particle is projected from a point O at a vertical distance a above a horizontal ground with initial velocity $\sqrt{ga}$ and at an angle $\alpha\left(0 < \alpha < \frac{\pi}{2}\right)$ to the horizontal, as shown in the figure. The particle strikes the ground at a horizontal distance a from O. Show that $\tan\alpha = 1 + \sqrt{2}$.



[see page three

3. A particle P of mass m is placed on a smooth horizontal table and is connected to a light smooth pulley Q by a light inextensible string which passes over a fixed small smooth pulley at the point A of the edge of the table. A light inextensible string which passes over the pulley Q is connected to particles of masses $2m$ and $3m$, as shown in the figure. The particles and the strings lie in a vertical plane. The system is released from rest with the strings taut. Obtain equations sufficient to determine the acceleration of Q.



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4. A car of mass M kg moves upwards with a constant acceleration along a straight road of inclination $\sin^{-1}\left(\frac{1}{20}\right)$ to the horizontal. There is a constant resistance of R N to its motion. The distance travelled by the car to increase its speed from 36 km h^{-1} to 72 km h^{-1} is 500 m. Obtain equations sufficient to determine the power exerted by the car when its speed is 54 km h^{-1}.

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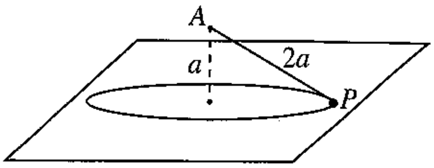
[see page four

5. One end of a light inextensible string of length $2a$ is attached to a fixed point A which is at a distance a vertically above a smooth horizontal table. A particle P of mass m, attached to the other end of the string, moves in a horizontal circle on the table with the string taut and with uniform speed $\sqrt{\dfrac{ga}{2}}$ (see the figure). Show that the magnitude of the normal reaction on the particle P from the table is $\dfrac{5}{6}mg$.

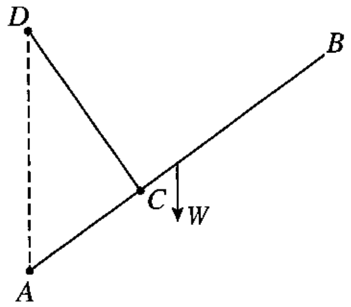


6. In the usual notation, the position vectors of two points A and B, with respect to a fixed origin O, are $2\mathbf{i}-3\mathbf{j}$ and $\mathbf{i}-2\mathbf{j}$, respectively. Using $\overrightarrow{AO}\cdot\overrightarrow{AB}$, find $O\hat{A}B$.

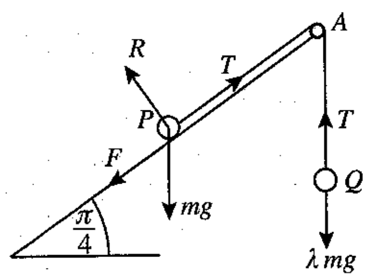
Let C be the point on OA such that $O\hat{C}B=\frac{\pi}{2}$. Find $\overrightarrow{OC}$.

[see page five

7. A uniform rod AB of length $8a$ and weight W has its end A smoothly hinged to a fixed point. One end of a light inextensible string of length $4a$ is attached to the point C on the rod such that $AC = 3a$, and the other end is attached to a fixed point D vertically above A such that $AD = 5a$ (see the figure). The rod is in equilibrium. Show that the tension of the string is $\frac{16}{15}W$.
Also, find the horizontal component of the reaction at A.



8. A particle P of mass m is placed on a rough plane inclined at an angle $\frac{\pi}{4}$ to the horizontal. One end of a light inextensible string which passes over a fixed small smooth pulley fixed to the edge of the inclined plane at A, is attached to the particle P and the other end to a particle Q of mass λmg, as shown in the figure. The coefficient of friction between the particle P and the inclined plane is $\frac{1}{2}$. The line PA is a line of greatest slope of the inclined plane and the particles P and Q stay in equilibrium with the string taut.
Show that $\frac{1}{2\sqrt{2}} \leq \lambda \leq \frac{3}{2\sqrt{2}}$. (The relevant forces are marked in the figure.)



9. Let A and B be two **independent** events of a sample space Ω. In the usual notation, it is given that $P(A)=\frac{1}{5}$ and $P(B)=\frac{3}{4}$. Find $P(A \cup B)$, $P(A \mid A \cup B)$ and $P(B \mid A')$, where A' denotes complementary event of A.

10. A set of five observations of positive integers has mean 6 and range 10. It has two modes. If the median is different from the modes, find the five observations.

අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, 2021(2022)
கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2021(2022)
General Certificate of Education (Adv. Level) Examination, 2021(2022)

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இணைந்த கணிதம் II
Combined Mathematics II

10 E II

Part B

* Answer **five** questions only.

(In this question paper, g denotes the acceleration due to gravity.)

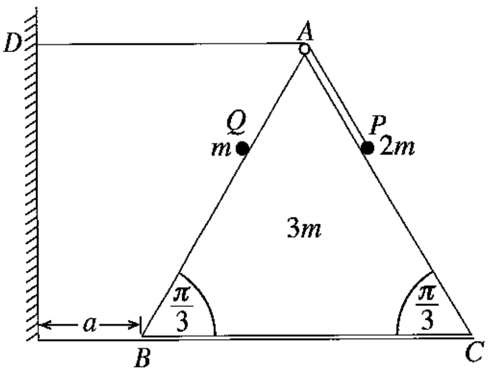
11. (*a*) A particle P, projected with a velocity $u \text{ m s}^{-1}$ vertically upwards from a point O, reaches a point A after 4 seconds and comes back to A again after another 2 seconds. At the instant when the particle P is at A for the second time, another particle Q is projected with the same velocity $u \text{ m s}^{-1}$ vertically upwards from O. Sketch the velocity-time graph for the motions of P and Q, in the same diagram.
Hence, find the value of u and the height of OA in terms of g, and the time taken by Q to collide with P.

(*b*) A ship S is sailing due north with uniform speed $u \text{ km h}^{-1}$ relative to earth. At a certain instant, a boat P is at a distance d km east of S and another boat Q is at a distance $\sqrt{3}\,d$ km south of S. The boat P travels in a straight line path intending to intercept S with uniform speed $2u \text{ km h}^{-1}$ relative to earth and the boat Q travels in a straight line path intending to intercept P with uniform speed $3u \text{ km h}^{-1}$ relative to earth.
Show that

(i) the time taken by the boat P to intercept the ship S is $\dfrac{d}{\sqrt{3}u}$ h,

(ii) the boat P intercepts the ship S before the boat Q intercepts the boat P.

12. (*a*) Equilateral triangle ABC in the figure is the vertical cross-section through the centre of gravity of a smooth uniform wedge of mass $3m$ with $AB = BC = AC = 6a$ such that the face containing BC is placed on a smooth horizontal floor. The lines AB and AC are lines of greatest slope of the faces containing those. The point D is a fixed point on the vertical wall which is at a distance a from the point B of the wedge, and in the plane of ABC such that AD is horizontal. One end of a light inextensible string of length $5a$ passing over a small smooth pulley fixed at A is attached to a particle P of mass $2m$ kept on AC and the other end is attached to the fixed point D on the wall. A particle Q of mass m is held on AB. The system is released from the rest with $AP = AQ = a$, as shown in the figure. Obtain equations sufficient to determine the velocity of Q relative to the wedge at the instant when the wedge strikes the wall.



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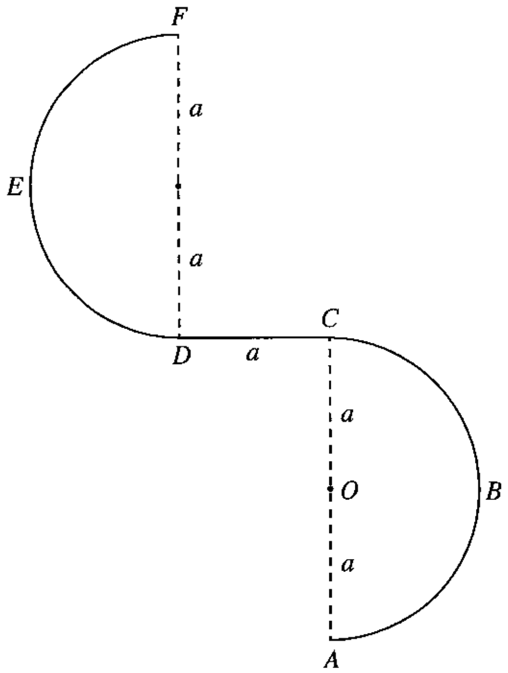
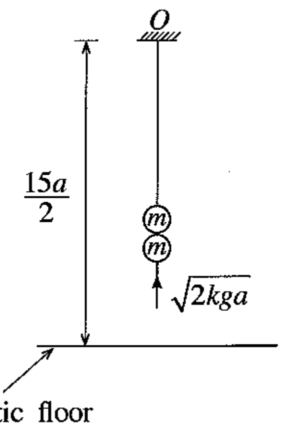
(*b*) A thin wire *ABCDEF* is fixed in a vertical plane, as shown in the figure. The portion *ABC* is a thin **smooth** semicircular wire with centre O and radius a. The portion *CD* is a thin **rough** horizontal wire of length a. The portion *DEF* is also a thin **smooth** semicircular wire of radius a. The diameters *AC* and *DF* are vertical. A small smooth bead P of mass m is placed at A and is given a velocity $u\left(>3\sqrt{ag}\right)$ horizontally, and begins to move along the wire. It is given that the magnitude of the frictional force on the bead from the wire, during its motion from C to D, is $\frac{1}{2}mg$. Show that the speed v of the bead P, during its motion from A to C, when $\overrightarrow{OP}$ makes an angle θ $(0 \leq \theta \leq \pi)$ with $\overrightarrow{OA}$, is given by $v^2 = u^2 - 2ag(1 - \cos\theta)$.



Show that the speed w of the bead P just before it leaves the wire at F is given by $w^2 = u^2 - 9ag$, and find the reaction on the bead P from the wire at that instant.

13. One end of a light elastic string of natural length $4a$ is attached to a fixed point O and the other end to a particle P of mass m. The particle hangs in equilibrium at a distance $5a$ below O. Show that the modulus of elasticity of the string is $4mg$.

Now, another particle Q of mass m moving vertically upwards collides and coalesces with P, and form a combined particle R. The speed of the particle Q just before it collides with the particle P is $\sqrt{2kga}$. Find the velocity with which R begins to move.



Show that, in the subsequent motion while the string is not slack, the distance x from O to the combined particle R satisfies the equation $\ddot{x}+\frac{g}{2a}(x-6a)=0$.

By writing $X = x - 6a$, show that, $\ddot{X} + w^2X = 0$ where $w = \sqrt{\frac{g}{2a}}$.

Find the centre of the above simple harmonic motion and using the formula $\dot{X}^2 = w^2(c^2 - X^2)$, find the amplitude c.

Show that if $k > 3$, then the string becomes slack,

Now, let $k = 8$. Find the time taken by the combined particle R to strike an **inelastic horizontal floor** at a distance $\frac{15}{2}a$ below the point O, from the instant of coalescing of the particles P and Q. Also, find the maximum height reached by the combined particle R after striking the floor.

14. (*a*) Let **a** and **b** be non-zero and non-parallel vectors, and $\lambda, \mu \in \mathbb{R}$.

Show that if $\lambda\mathbf{a} + \mu\mathbf{b} = \mathbf{0}$, then $\lambda = 0$ and $\mu = 0$.

Let ABC be a triangle. The mid-point of AB is D and the mid-point of CD is E. The lines AE (extended) and BC meet at F. Let $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AC} = \mathbf{b}$. Using the triangle law of addition, show that $\overrightarrow{AE} = \dfrac{\mathbf{a}+2\mathbf{b}}{4}$.

Explain why $\overrightarrow{AF} = \alpha\overrightarrow{AE}$ and $\overrightarrow{CF} = \beta\overrightarrow{CB}$, where $\alpha, \beta \in \mathbb{R}$.

Considering the triangle ACF, show that $(\alpha - 4\beta)\mathbf{a} + 2(\alpha + 2\beta - 2)\mathbf{b} = \mathbf{0}$.

Hence, find the values of α and β.

(*b*) Let ABC be an equilateral triangle of sides $2a$ and let D, E, F be the mid points of AB, BC and AC respectively. Forces of magnitudes $2P$, $\sqrt{3}\,P$, $2\sqrt{3}\,P$ and αP act respectively along $\overrightarrow{AB}$, $\overrightarrow{AE}$, $\overrightarrow{DC}$ and $\overrightarrow{BC}$. It is given that the resultant of this system of forces is acting parallel to $\overrightarrow{AC}$. Find the value of α.

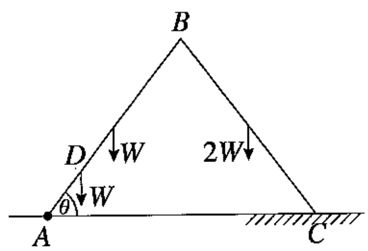
The system of forces is equivalent to a single force of magnitude R acting through A together with a couple of magnitude G. Find the values of R and G.

Write down the magnitude and the direction of the resultant of this system of forces and find the distance from A to the point at which the line of action of the resultant meets AB.

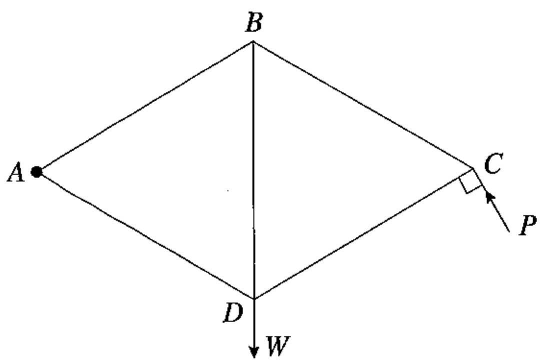
A couple of magnitude H is now added to the system. The resultant of this new system acts through the point B. Find the value of H and the sense of this couple.

15. (*a*) Two uniform rods AB and BC, each of length $2a$, are smoothly joined at the end B. The weights of the rods AB and BC are W and $2W$, respectively. The end A is smoothly hinged to a fixed point on a horizontal floor. A particle of weight W is attached to the point D on rod AB such that $AD = \frac{a}{2}$. The system is in equilibrium in a vertical plane such that $B\hat{A}C = \theta$ and the end-point C of the rod BC on a rough portion of the above horizontal floor, as shown in the figure. The coefficient of friction between the rod BC and the floor is μ. Show that $\cot\theta \le \frac{15}{7}\mu$.

Find the reaction exerted on AB by CB at the joint B.



(*b*) The framework shown in the figure consists of five light rods AB, BC, CD, DA and DB of equal lengths smoothly jointed at their ends. A load W is suspended at the joint D and the framework is smoothly hinged at A to a fixed point and kept in equilibrium in a vertical plane with BD vertical by a force P applied to it at the joint C and perpendicular to the rod CD, in the direction shown in the figure.



(i) Find the value of P.

(ii) Draw a stress diagram using Bow's notation for the joints C, B and D.

Hence, find the stresses in the rods, stating whether they are tensions or thrusts.

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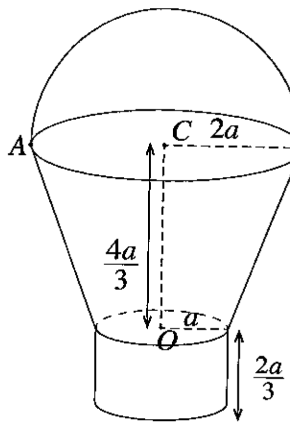
16. Show that the centre of mass of

(i) a thin uniform wire in the shape of a semi-circular arc of radius a, is at a distance $\frac{2a}{\pi}$ from its centre,

(ii) a uniform hollow right circular cone of height h is at a distance $\frac{1}{3}h$ from the centre of the base of the cone.

A bucket is made by rigidly fixing to a uniform thin shell in the shape of a frustum of hollow right circular cone of radii of the upper and lower circular rims $2a$ and a, respectively and height $\frac{4a}{3}$ the following parts at the places each meets this shell as shown in the figure:



- A uniform thin circular plate of radius a and centre at O.
- A uniform thin shell in the shape of a hollow right circular cylinder of radius a and height $\frac{2a}{3}$.
- A uniform thin wire in the shape of a semi-circle of radius $2a$ and centre at C.

The mass per unit area of the frustum, plate and the cylinder is σ and the mass per unit length of the wire is $11a\sigma$.

Show that the distance from O to the centre of mass of the bucket is $(10\pi+27)\frac{a}{9\pi}$.

Find the angle OC makes with the downward vertical in the equilibrium position, when the bucket is hanged freely by a vertical string from the point A at which the wire meets the upper rim of the frustum.

17.(*a*) Two identical boxes A and B, each contains 10 balls which are identical in all respects except for their colour. The box A contains 6 white balls and 4 red balls, and the box B contains 8 white balls and 2 red balls. A box is chosen at random and 3 balls are drawn from that box at random, one after the other, without replacement. Find the probability that

(i) two red balls and one white ball are drawn,

(ii) the box A was chosen, given that two red balls and one white ball are drawn.

(*b*) Let the mean and the standard deviation of the set of data $\{x_1, x_2, \ldots, x_n\}$ be $\bar{x}$ and σ_x respectively, and let $y_i = \frac{x_i - \alpha}{\beta}$ for $i = 1, 2, \ldots, n$ where α and β (>0) are real constants. Show that $\bar{y} = \frac{\bar{x} - \alpha}{\beta}$ and $\sigma_y = \frac{\sigma_x}{\beta}$, where $\bar{y}$ and σ_y are respectively the mean and the standard deviation of the set of data $\{y_1, y_2, \ldots, y_n\}$.

Monthly instalments for an insurance scheme by 100 employees of a company are given in the following frequency table:

Monthly Instalment (rupees) x	Number of employees
1500 – 3500	30
3500 – 5500	40
5500 – 7500	20
7500 – 9500	10

By means of the transformation $y = \frac{x - 500}{1000}$, estimate the mean and the standard deviation of y, and also the coefficient of skewness of y defined by $\frac{3(\text{mean} - \text{median})}{\text{standard deviation}}$.

Hence, estimate the mean, the standard deviation and the coefficient of skewness of x.

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