Confidential

Department of Examinations - Sri Lanka

G.C.E. (A/L) Examination - 2017

10 - Combined Mathematics I

Marking Scheme

This has been prepared for the use of marking examiners. Changes would be made according to the views presented at the Chief/Assistant Examiners' meeting.

G.C.E. (A/L) Examination - 2017

10 - Combined Mathematics

Distribution of Marks

Paper I :

Part A :	10 x 25	=	250
Part B :	05 x 150	=	750

Total **= 1000/10**

Paper 1- Final Mark **= 100**

1. Using the **Principle of Mathematical Induction**, prove that $\sum_{r=1}^{n} r(3r+1) = n(n+1)^2$ for all $n \in \mathbb{Z}^+$.

For $n=1$, L.H.S. $= 1\cdot(3+1) = 4$ and R.H.S. $= 1\cdot(1+1)^2 = 4$. (5)

$\therefore$ The result is true for $n=1$.

Take any $p \in \mathbb{Z}^+$ and assume that the result is true for $n = p$.

i.e. $\sum_{r=1}^{p} r(3r+1) = p(p+1)^2$. --------------- (1) (5)

Now $\sum_{r=1}^{p+1} r(3r+1) = \sum_{r=1}^{p} r(3r+1) + (p+1)(3p+4)$ (5)

$= p(p+1)^2 + (p+1)(3p+4)$

$= (p+1)(p^2 + p + 3p + 4)$

$= (p+1)(p+2)^2$. (5)

Hence if the result is true for $n = p$, then it is also true for $n = p+1$. We have already proved that the result is true for $n=1$.

Hence by the Principle of Mathematical Induction, the result is true for all $n \in \mathbb{Z}^+$.

(5)

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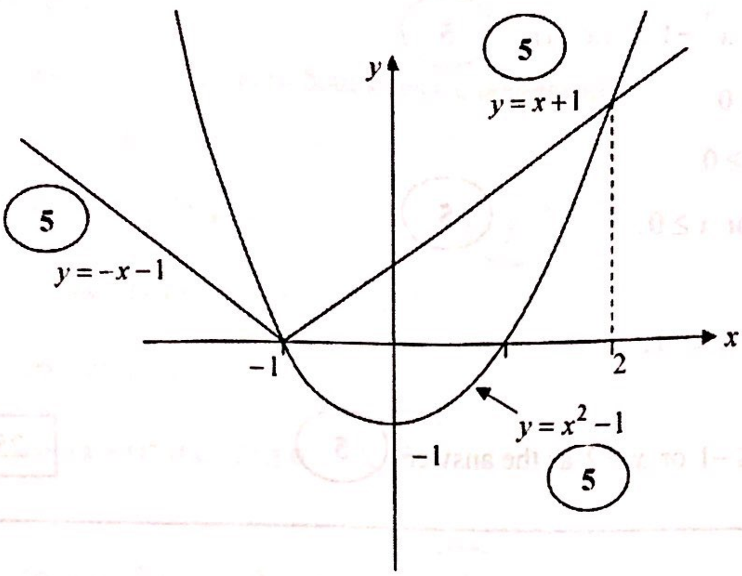
2. Find all real values of x satisfying the inequality $x^2 - 1 \geq |x + 1|$.



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At the points of intersection, we must have $x \geq -1$ and $x^2 - 1 = x + 1$, and so $x = -1$ or $x = 2$.

The solutions are the values of x satisfying $x \leq -1$ or $x \geq 2$. (5)

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Aliter 1

$$|x+1| = \begin{cases} x+1 & \text{if } x \geq -1 \\ -(x+1) & \text{if } x < -1 \end{cases}$$

Case (i) $x \geq -1$

In this case, $x^2 - 1 \geq |x+1| \Leftrightarrow x^2 - 1 \geq x + 1$ (5)

$\Leftrightarrow x^2 - x - 2 \geq 0$

$\Leftrightarrow (x+1)(x-2) \geq 0$

$\Leftrightarrow x \leq -1$ or $x \geq 2$. (5)

Since $x \geq -1$, the solutions are $x = -1$ or $x \geq 2$.

Case (ii) $x < -1$,

In this case, $x^2 - 1 \geq |x+1| \Leftrightarrow x^2 - 1 \geq -(x+1)$ (5)

$\Leftrightarrow x^2 + x \geq 0$

$\Leftrightarrow x(x+1) \geq 0$

$\Leftrightarrow x \leq -1$ or $x \geq 0$. (5)

Since $x < -1$, the solutions are $x < -1$.

From the two cases, we get $x \leq -1$ or $x \geq 2$ as the answer. (5)

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Aliter 2

Case (i) $x > -1$

$x^2 - 1 \geq |x+1| \Leftrightarrow x^2 - 1 \geq x + 1$ (5)

$\Leftrightarrow x \leq -1$ or $x \geq 2$. (5)

Since $x > -1$, the solutions are $x \geq 2$.

Case (ii) $x \leq -1$

$x^2 - 1 \geq |x+1| \Leftrightarrow x^2 - 1 \geq -(x+1)$ (5)

$\Leftrightarrow x \leq -1$ or $x \geq 0$. (5)

Since $x \leq -1$, the solutions are $x \leq -1$.

From the two cases, we get $x \leq -1$ or $x \geq 2$ as the answer. (5)

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Aliter 3

Case (i) $x^2 \geq 1$

In this case $x^2 - 1 \geq 0$, and so both sides are non-negative.

$\therefore\ x^2 - 1 \geq |x+1|$

$\Leftrightarrow (x^2-1)^2 \geq (x+1)^2$ (5)

$\Leftrightarrow (x+1)^2(x-1)^2 - (x+1)^2 \geq 0$

$\Leftrightarrow (x+1)^2[(x-1)^2 - 1] \geq 0$

$\Leftrightarrow (x+1)^2 x(x-2) \geq 0$ (5)

$\Leftrightarrow x = -1$ or $x \leq 0$ or $x \geq 2$ (5)

Since $x^2 \geq 1 \Leftrightarrow x \leq -1$ or $x \geq 1$, the solutions are $x \leq -1$ or $x \geq 2$. (5)

Case (ii) $x^2 < 1$

Since $x^2 - 1 < 0$, and hence there are no solution. From the two cases, we get $x \leq -1$ or $x \geq 2$ as the answer. (5)

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5. Let $0 < \alpha < \frac{\pi}{2}$. Show that $\lim_{x \to \alpha} \frac{x^3 - \alpha^3}{\tan x - \tan \alpha} = 3\alpha^2 \cos^2 \alpha$.

(5)

$$\lim_{x\to\alpha} \frac{x^3-\alpha^3}{\tan x - \tan\alpha} = \lim_{x\to\alpha} \frac{(x-\alpha)(x^2+\alpha x+\alpha^2)}{\frac{\sin x}{\cos x} - \frac{\sin\alpha}{\cos\alpha}}$$

$$= \lim_{x\to\alpha} \frac{(x-\alpha)\cos x\cos\alpha\cdot(x^2+\alpha x+\alpha^2)}{\sin x\cos\alpha - \cos x\sin\alpha}$$ (5)

$$= \lim_{x\to\alpha} \frac{x-\alpha}{\sin(x-\alpha)}\cdot\cos x\cos\alpha\cdot(x^2+\alpha x+\alpha^2)$$ (5)

(5)

$$= 1\cdot\cos\alpha\cdot\cos\alpha\cdot(3\alpha^2)$$

$$= 3\alpha^2\cos^2\alpha.$$ (5)

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Aliter 1

(5)

$$\lim_{x\to\alpha} \frac{x^3-\alpha^3}{\tan x - \tan\alpha} = \lim_{x\to\alpha} \frac{(x-\alpha)(x^2+\alpha x+\alpha^2)}{\tan(x-\alpha)(1+\tan x\tan\alpha)}$$ $\left(\because \tan(x-\alpha) = \frac{\tan x - \tan\alpha}{1+\tan x\tan\alpha}\right)$

$$= \lim_{x\to\alpha} \frac{x-\alpha}{\tan(x-\alpha)}\cdot\frac{x^2+\alpha x+\alpha^2}{(1+\tan x\tan\alpha)}$$ (5)

$$= \lim_{x\to\alpha} \frac{x-\alpha}{\sin(x-\alpha)}\cdot\frac{\cos(x-\alpha)\cdot(x^2+\alpha x+\alpha^2)}{(1+\tan x\tan\alpha)}$$

(5)

$$= 1\cdot\frac{1\cdot 3\alpha^2}{1+\tan^2\alpha}$$ (5)

$$= \frac{3\alpha^2}{\sec^2\alpha} = 3\alpha^2\cos^2\alpha.$$ (5)

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Aliter 2

$$\lim_{x\to\alpha}\frac{x^3-\alpha^3}{\tan x-\tan\alpha} = \lim_{x\to\alpha}\frac{x^3-\alpha^3}{x-\alpha}\cdot\frac{x-\alpha}{\frac{\sin x}{\cos x}-\frac{\sin\alpha}{\cos\alpha}}$$ (5)

$$= \lim_{x\to\alpha}\frac{x^3-\alpha^3}{x-\alpha}\cdot\frac{x-\alpha}{\frac{\sin x\cos\alpha-\cos x\sin\alpha}{\cos x\cos\alpha}}$$ (5)

$$= \lim_{x\to\alpha}\frac{x^3-\alpha^3}{x-\alpha}\cdot\frac{(x-\alpha)}{\sin(x-\alpha)}\cdot\cos x\cos\alpha$$ (5)

$$=3\alpha^2\cdot 1\cdot\cos^2\alpha$$

(5)

$$=3\alpha^2\cos^2\alpha\,.$$ (5) [25]

6. Let $0<a<b$. Show that $\frac{d}{dx}\sin^{-1}\left(\sqrt{\frac{b-a}{b}}\cos x\right)=-\frac{\sqrt{b-a}\sin x}{\sqrt{a\cos^2x+b\sin^2x}}$.

Hence, find $\int\frac{\sin x}{\sqrt{a\cos^2x+b\sin^2x}}dx$.

$$\frac{d}{dx}\sin^{-1}\left(\sqrt{\frac{b-a}{b}}\cos x\right)=\frac{1}{\sqrt{1-\frac{(b-a)}{b}\cos^2x}}\times\sqrt{\frac{b-a}{b}}\times(-\sin x)$$ (5) + (5)

$$=-\frac{\sin x}{\sqrt{b-b\cos^2x+a\cos^2x}}\times\sqrt{b-a}$$

$$=-\frac{\sqrt{b-a}\sin x}{\sqrt{a\cos^2x+b\sin^2x}}$$ (5)

$$\therefore\int-\frac{\sqrt{b-a}\sin x}{\sqrt{a\cos^2x+b\sin^2x}}dx=\sin^{-1}\left(\sqrt{\frac{b-a}{b}}\cos x\right)+\text{constant}$$ (5)

$$\int \frac{\sin x}{\sqrt{a\cos^2 x + b\sin^2 x}}\,dx = -\frac{1}{\sqrt{b-a}}\sin^{-1}\left(\sqrt{\frac{b-a}{b}}\cos x\right) + C,$$ where C is an arbitrary constant. (5)

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Aliter

Let $y = \sin^{-1}\left(\sqrt{\frac{b-a}{b}}\cos x\right)$.

Then $\sin y = \sqrt{\frac{b-a}{b}}\cos x$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

$$\cos y\frac{dy}{dx} = \sqrt{\frac{b-a}{b}}(-\sin x) \text{ ---------------- (1)} \quad (5)$$

$$\cos y = \sqrt{1-\sin^2 y} \quad \left(\because -\frac{\pi}{2} \le y \le \frac{\pi}{2}\right)$$

$$= \sqrt{1 - \frac{b-a}{b}\cos^2 x}$$

$$= \sqrt{\frac{b(1-\cos^2 x) + a\cos^2 x}{b}}$$

$$= \frac{\sqrt{a\cos^2 x + b\sin^2 x}}{\sqrt{b}} \quad (5)$$

$$\therefore (1) \Rightarrow \frac{dy}{dx} = -\frac{\sqrt{b-a}\sin x}{\sqrt{a\cos^2 x + b\sin^2 x}}. \quad (5)$$

Integration as before. (10)

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7. A curve C is given parametrically by $x=3\cos\theta-\cos^3\theta$, $y=3\sin\theta-\sin^3\theta$ for $0<\theta<\frac{\pi}{2}$.
Show that $\frac{dy}{dx}=-\cot^3\theta$.
Find the coordinates of the point P on the curve C at which the gradient of the tangent line is -1.

$x=3\cos\theta-\cos^3\theta$ $\qquad$ $y=3\sin\theta-\sin^3\theta$

$\frac{dx}{d\theta}=-3\sin\theta+3\cos^2\theta\sin\theta$; (5) $\quad \frac{dy}{d\theta}=3\cos\theta-3\sin^2\theta\cos\theta$ (5)

$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{3\cos\theta(1-\sin^2\theta)}{-3\sin\theta(1-\cos^2\theta)}=-\frac{\cos^3\theta}{\sin^3\theta}=-\cot^3\theta.$$ (5)

$\frac{dy}{dx}=-1 \Leftrightarrow \cot\theta=1 \Leftrightarrow \theta=\frac{\pi}{4}$ (5)

$$P\equiv\left(\frac{3}{\sqrt{2}}-\frac{1}{2\sqrt{2}},\frac{3}{\sqrt{2}}-\frac{1}{2\sqrt{2}}\right)\equiv\left(\frac{5}{2\sqrt{2}},\frac{5}{2\sqrt{2}}\right).$$ (5)

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8. Let l_1 and l_2 be the straight lines given by $3x-4y=2$ and $4x-3y=1$ respectively.
(i) Write down the equations of the bisectors of the angles between l_1 and l_2.
(ii) Find the equation of the bisector of the acute angle between l_1 and l_2.

Bisectors are given by

$$\frac{3x-4y-2}{5}=\pm\frac{4x-3y-1}{5}$$ (5)

$x+y+1=0$ or $7x-7y-3=0$ (5)

Let α be the acute angle between l_1 and $x+y+1=0$

$$\tan\alpha=\left|\frac{\frac{3}{4}+1}{1-\frac{3}{4}}\right|$$ (5)

$=7>1$. (5)

$\therefore 7x-7y-3=0$ is the bisector of the acute angle between l_1 and l_2. (5)

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9. Let S be the circle given by $x^2 + y^2 - 4 = 0$ and let l be the straight line given by $y = x + 1$. Find the equation of the circle which passes through the points of intersection of S and l, and also intersects the circle S orthogonally.

The required equation has the form $(x^2+y^2-4)+\lambda(y-x-1)=0$, where $\lambda \in \mathbb{R}$.

i.e. $x^2+y^2-\lambda x+\lambda y-\lambda-4=0$. (10)

If this is orthogonal to S, with $g=0; f=0; c=-4; g'=-\dfrac{\lambda}{2}; f'=\dfrac{\lambda}{2}; c'=-\lambda-4,$

we must have $2gg'+2ff'=c+c'$. (5)

i.e. $0=-\lambda-8$

$\therefore \lambda=-8$. (5)

$\therefore$ The answer is $x^2+y^2+8x-8y+4=0$. (5)

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10. Show that $\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)^2 = 1+\sin\theta$ for $-\pi<\theta\le\pi$. Hence, show that $\cos\frac{\pi}{12}+\sin\frac{\pi}{12}=\sqrt{\frac{3}{2}}$ and also find the value of $\cos\frac{\pi}{12}-\sin\frac{\pi}{12}$. Deduce that $\sin\frac{\pi}{12}=\frac{\sqrt{3}-1}{2\sqrt{2}}$.

$$\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^2=\sin^2\frac{\theta}{2}+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}+\cos^2\frac{\theta}{2}$$ (5)

$$=1+\sin\theta \qquad (\because \sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}=1 \text{ and } 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}=\sin\theta.)$$

Let $\theta=\dfrac{\pi}{6}$: (5)

Then $\left(\cos\dfrac{\pi}{12}+\sin\dfrac{\pi}{12}\right)^2=1+\dfrac{1}{2}$.

$\therefore \sin\dfrac{\pi}{12}+\cos\dfrac{\pi}{12}=\sqrt{\dfrac{3}{2}}$ ------------- (1) (5) $\quad (\because \sin\dfrac{\pi}{12}+\cos\dfrac{\pi}{12}>0)$

Let $\theta = \frac{-\pi}{6}$:

Then $\left(\cos\frac{\pi}{12} - \sin\frac{\pi}{12}\right)^2 = \frac{1}{2}$.

$\therefore \cos\frac{\pi}{12} - \sin\frac{\pi}{12} = \frac{1}{\sqrt{2}}$ ------------- (2) ($\because \sin\frac{\pi}{12} < \cos\frac{\pi}{12}$) (5)

$(1) - (2) \Rightarrow \sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$. (5)

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Part B

11. (*a*) Let $f(x) = 3x^2 + 2ax + b$, where $a, b \in \mathbb{R}$.

It is given that the equation $f(x) = 0$ has two real distinct roots. Show that $a^2 > 3b$.

Let α and β be the roots of $f(x)=0$. Write down $\alpha+\beta$ in terms of a and $\alpha\beta$ in terms of b.

Show that $|\alpha - \beta| = \frac{2}{3}\sqrt{a^2 - 3b}$.

Show further that the quadratic equation with $|\alpha + \beta|$ and $|\alpha - \beta|$ as its roots is given by $9x^2 - 6\left(|a| + \sqrt{a^2 - 3b}\right)x + 4\sqrt{a^4 - 3a^2 b} = 0$.

(*b*) Let $g(x) = x^3 + px^2 + qx + 1$, where $p, q \in \mathbb{R}$. When $g(x)$ is divided by $(x-1)(x+2)$, the remainder is $3x+2$. Show that the remainder when $g(x)$ is divided by $(x-1)$ is 5, and that the remainder when $g(x)$ is divided by $(x+2)$ is -4.

Find the values of p and q, and show that $(x+1)$ is a factor of $g(x)$.

(5)

(a) The discriminant $\Delta = (2a)^2 - 4(3)(b)$

$= 4(a^2 - 3b)$. (5)

Since $f(x) = 0$ has two real distant roots, we must have $\Delta > 0$. (5)

$\therefore \; a^2 > 3b$. (5) **20**

$\alpha + \beta = -\frac{2a}{3}$ and $\alpha\beta = \frac{b}{3}$.

(5) (5) **10**

$(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta$ (10)

$=\frac{4a^2}{9}-\frac{4b}{3}$ (5)

$=\frac{4}{9}(a^2-3b).$ (5)

$\therefore\ |\alpha-\beta|=\frac{2}{3}\sqrt{a^2-3b}.$ (5)

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Let $\alpha'=|\alpha+\beta|$ and $\beta'=|\alpha-\beta|$.

Then $\alpha'=\frac{2}{3}|a|$ and $\beta'=\frac{2}{3}\sqrt{a^2-3b}.$ (5)

The required equation is $(x-\alpha')(x-\beta')=0.$ (5)

i.e. $x^2-(\alpha'+\beta')x+\alpha'\beta'=0.$ (5)

$\Rightarrow x^2-\left(\frac{2}{3}|a|+\frac{2}{3}\sqrt{a^2-3b}\right)x+\frac{4}{9}|a|\sqrt{a^2-3b}=0.$ (5) (5)

$\Rightarrow 9x^2-6\left(|a|+\sqrt{a^2-3b}\right)x+4\sqrt{a^4-3a^2b}=0.$ (5)

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(b) Since the remainder when $g(x)$ divided by $(x-1)(x+2)$ is $3x+2$, we have

$g(x)=h(x)(x-1)(x+2)+3x+2,$ ------------ (1) (10)

where $h(x)$ is a polynomial of degree 1.

By the Remainder Theorem, the remainder when $g(x)$ is divided by $(x-1)$ is $g(1)$.

(5)

$(1) \Rightarrow g(1) = 5.$ (5)

Hence, the remainder when $g(x)$ divided by $(x-1)$ is 5.

Again, by the Remainder Theorem, the remainder when $g(x)$ is divided by $(x+2)$ is $g(-2)$. (5)

$(1) \Rightarrow g(-2) = -4.$ (5)

Hence, the remainder when $g(x)$ divided by $(x+2)$ is -4.

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$g(1) = 5 \Rightarrow 1 + p + q + 1 = 5$ (5)

$p + q = 3$

$g(-2) = -4 \Rightarrow -8 + 4p - 2q + 1 = -4$ (5)

$4p - 2q = 3$

$p = \frac{3}{2}$ and $q = \frac{3}{2}$.

(5) (5)

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(5) (5)

Now $g(-1) = -1 + p - q + 1 = 0. (\because p = q)$

Thus, by the Factor Theorem, $(x+1)$ is a factor of $g(x)$. (5)

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12.(a) Write down the binomial expansion of $(5+2x)^{14}$ in ascending powers of x.

Let T_r be the term containing x^r in the above expansion for $r = 0, 1, 2, \ldots, 14$.

Show that $\frac{T_{r+1}}{T_r} = \frac{2(14-r)}{5(r+1)}x$ for $x \neq 0$.

Hence, find the value of r which gives the largest term of the above expansion, when $x = \frac{4}{3}$.

(b) Let $c \geq 0$. Show that $\frac{2}{(r+c)(r+c+2)} = \frac{1}{(r+c)} - \frac{1}{(r+c+2)}$ for $r \in \mathbb{Z}^+$.

Hence, show that $\sum_{r=1}^{n} \frac{2}{(r+c)(r+c+2)} = \frac{(3+2c)}{(1+c)(2+c)} - \frac{1}{(n+c+1)} - \frac{1}{(n+c+2)}$ for $n \in \mathbb{Z}^+$.

Deduce that the infinite series $\sum_{r=1}^{\infty} \frac{2}{(r+c)(r+c+2)}$ converges and find its sum.

By using this sum with suitable values for c, show that $\sum_{r=1}^{\infty} \frac{1}{r(r+2)} = \frac{1}{3} + \sum_{r=1}^{\infty} \frac{1}{(r+1)(r+3)}$.

(a) $(5+2x)^{14} = \sum_{r=0}^{14} {}^{14}C_r 5^{14-r}(2x)^r$ (10)

$= \sum_{r=0}^{14} {}^{14}C_r 5^{14-r} \cdot 2^r \cdot x^r$, where ${}^{14}C_r = \frac{14!}{r!(14-r)!}$ for $r = 0, 1, \ldots, 14$.

(5)

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Let $T_r = {}^{14}C_r 5^{14-r} \cdot 2^r \cdot x^r$ for $r = 0, 1, \ldots, 14$.

Then $\frac{T_{r+1}}{T_r} = \frac{14!\, 5^{13-r}\, 2^{r+1}}{(r+1)!(13-r)!} x^{r+1} \Big/ \frac{14!\, 5^{14-r}\, 2^r}{r!(14-r)!} x^r$ (5)

(10)

$= \frac{2(14-r)}{5(r+1)} x$ (5)

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Thus. $x=\frac{4}{3} \Rightarrow \frac{T_{r+1}}{T_r}=\frac{2}{5}\frac{(14-r)}{(r+1)}\cdot\frac{4}{3}$ (5)

Hence, $\frac{T_{r+1}}{T_r} \gtreqless 1$ according as $\frac{8}{15}\frac{(14-r)}{(r+1)} \gtreqless 1$.

(5) (5)

i.e. according as $112-8r \gtreqless 15r+15$.

i.e. according as $r \lesseqgtr \frac{97}{23}=4\frac{5}{23}$.

(5)

$T_0 < T_1 < T_2 < T_3 < T_4 < T_5 > T_6 \cdots > T_{14}$ (10)

$\therefore$ The required value is $r=5$. (5)

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(b) $\frac{1}{r+c}-\frac{1}{r+c+2}=\frac{(r+c+2)-(r+c)}{(r+c)(r+c+2)}$ (5)

$=\frac{2}{(r+c)(r+c+2)}$. (5)

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Let $u_r=\frac{2}{(r+c)(r+c+2)}$ for $r \in \mathbb{Z}^+$

Then

$r=1;\quad u_1=\frac{1}{1+c}-\frac{1}{3+c}$

(5)

$r=2;\quad u_2=\frac{1}{2+c}-\frac{1}{4+c}$

$r=3;\quad u_3=\frac{1}{3+c}-\frac{1}{5+c}$ (5)

$\vdots$

$$r = n-2;\; u_{n-2} = \frac{1}{n-2+c} - \frac{1}{n+c}$$ (5)

$$r = n-1;\; u_{n-1} = \frac{1}{n-1+c} - \frac{1}{n+c+1}$$

$$r = n;\; u_n = \frac{1}{n+c} - \frac{1}{n+c+2}$$ (5)

$$\sum_{r=1}^{n} u_r = \frac{1}{1+c} + \frac{1}{2+c} - \frac{1}{n+c+1} - \frac{1}{n+c+2}$$ (10)

$$= \frac{3+2c}{(1+c)(2+c)} - \frac{1}{n+c+1} - \frac{1}{n+c+2}$$ (5)

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The limit of the R.H.S. as $n \to \infty$ is $\frac{3+2c}{(1+c)(2+c)}$. (10)

$\therefore\ \sum_{r=1}^{\infty} u_r$ convergent and the sum is $\frac{3+2c}{(1+c)(2+c)}$.

(5) (5)

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Put $c = 0$: $\sum_{r=1}^{\infty} \frac{1}{r(r+2)} = \frac{3}{4}$. ----------------(1) (5)

Put $c = 1$: $\sum_{r=1}^{\infty} \frac{1}{(r+1)(r+3)} = \frac{5}{12}$. (5)

$\Rightarrow \frac{1}{3} + \sum_{r=1}^{\infty} \frac{1}{(r+1)(r+3)} = \frac{1}{3} + \frac{5}{12} = \frac{3}{4}$ ----------------(2)

Now, (1) and (2) $\Rightarrow \sum_{r=1}^{\infty} \frac{1}{r(r+2)} = \frac{1}{3} + \sum_{r=1}^{\infty} \frac{1}{(r+1)(r+3)}$. (5)

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13.(a) Let $\mathbf{A}=\begin{pmatrix}2 & a & 3\\ -1 & b & 2\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}1 & -1 & a\\ 1 & b & 0\end{pmatrix}$ and $\mathbf{P}=\begin{pmatrix}4 & 1\\ 2 & 0\end{pmatrix}$, where $a, b \in \mathbb{R}$.

It is given that $\mathbf{AB}^{\mathrm{T}}=\mathbf{P}$, where $\mathbf{B}^{\mathrm{T}}$ denotes the transpose of the matrix $\mathbf{B}$. Show that $a=1$ and $b=-1$, and with these values for a and b, find $\mathbf{B}^{\mathrm{T}}\mathbf{A}$.

Write down $\mathbf{P}^{-1}$, and using it, find the matrix $\mathbf{Q}$ such that $\mathbf{PQ}=\mathbf{P}^2+2\mathbf{I}$, where $\mathbf{I}$ is the identity matrix of order 2.

(b) Sketch in an Argand diagram, the locus C of the points representing complex numbers z satisfying $|z|=1$.

Let $z_0=a(\cos\theta+i\sin\theta)$, where $a>0$ and $0<\theta<\frac{\pi}{2}$. Find the modulus in terms of a and the principal argument, in terms of θ, of each of the complex numbers $\frac{1}{z_0}$ and z_0^2.

Let P, Q, R and S be the points in the above Argand diagram representing the complex numbers z_0, $\frac{1}{z_0}$, $z_0+\frac{1}{z_0}$ and z_0^2, respectively.

Show that when the point P lies on C above,

(i) the points Q and S also lie on C, and

(ii) the point R lies on the real axis between 0 and 2.

(a) $$AB^T=\begin{pmatrix}2 & a & 3\\ -1 & b & 2\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & b\\ a & 0\end{pmatrix} \quad (5)$$

$$=\begin{pmatrix}2-a+3a & 2+ab\\ -1-b+2a & -1+b^2\end{pmatrix} \quad (10)$$

$$AB^T=P \Leftrightarrow \begin{pmatrix}2-a+3a & 2+ab\\ -1-b+2a & -1+b^2\end{pmatrix}=\begin{pmatrix}4 & 1\\ 2 & 0\end{pmatrix} \quad (5)$$

$$\Leftrightarrow\ 2+2a=4,\quad 2+ab=1,\quad -1+2a-b=2,\quad -1+b^2=0. \quad (10)$$

$$\Leftrightarrow\ a=1,\qquad b=-1. \quad (5)$$

$$\text{Now, } B^T A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 & 3 \\ -1 & -1 & 2 \end{pmatrix} \quad (5)$$

$$= \begin{pmatrix} 1 & 0 & 5 \\ -1 & 0 & -5 \\ 2 & 1 & 3 \end{pmatrix}. \quad (5)$$

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$$P^{-1} = -\frac{1}{2}\begin{pmatrix} 0 & -1 \\ -2 & 4 \end{pmatrix}. \quad (10)$$

$$\text{Also, } PQ = P^2 + 2I \Leftrightarrow P^{-1}(PQ) = P^{-1}(P^2 + 2I) \quad (5)$$

$$\Leftrightarrow Q = P^{-1}P^2 + P^{-1}(2I) \quad (5)$$

$$\Leftrightarrow Q = P + 2P^{-1} \quad (5)$$

$$\Leftrightarrow Q = \begin{pmatrix} 4 & 1 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 2 & -4 \end{pmatrix} \quad (5)$$

$$\therefore Q = \begin{pmatrix} 4 & 2 \\ 4 & -4 \end{pmatrix}. \quad (5)$$

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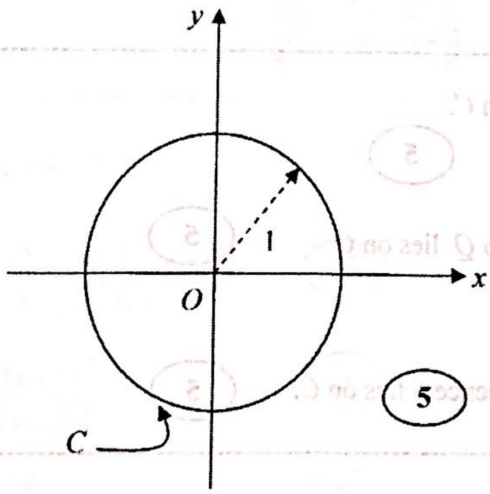
(b)



(5)

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First, $\dfrac{1}{z_0} = \dfrac{1}{a(\cos\theta + i\sin\theta)} \cdot \dfrac{(\cos\theta - i\sin\theta)}{(\cos\theta - i\sin\theta)}$ (5)

$= \dfrac{(\cos\theta - i\sin\theta)}{a(\cos^2\theta + \sin^2\theta)}$

$= \dfrac{1}{a}(\cos(-\theta) + i\sin(-\theta))$ (5)

Hence, $\left|\dfrac{1}{z_0}\right| = \dfrac{1}{a}$, and $\text{Arg}\left(\dfrac{1}{z_0}\right) = -\theta.$

(5) (5)

Next, $z_0{}^2 = a^2(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)$

$= a^2\left\{(\cos^2\theta - \sin^2\theta) + 2i\cos\theta\sin\theta\right\}$ (5)

$= a^2(\cos 2\theta + i\sin 2\theta)$ (5)

Hence, $\left|z_0{}^2\right| = a^2$, and $\text{Arg}\left(z_0{}^2\right) = 2\theta.$ (5)

(5) **40**

Suppose that P lies on C.

Then $a = 1.$ (5)

$\therefore \left|\dfrac{1}{z_0}\right| = 1$ and so Q lies on C (5)

Also, $|z_0^2| = 1$ and hence S lies on C. (5)

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$z_0 + \dfrac{1}{z_0} = (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta)$

$= 2\cos\theta.$ (5)

Note that $0 < \theta < \frac{\pi}{2} \Rightarrow 0 < 2\cos\theta < 2$.

$\therefore$ The number represented by $z_0 + \frac{1}{z_0}$ is real and lies between 0 and 2 on the real axis. (5)

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14. *(a)* Let $f(x) = \frac{x^2}{(x-1)(x-2)}$ for $x \neq 1, 2$.

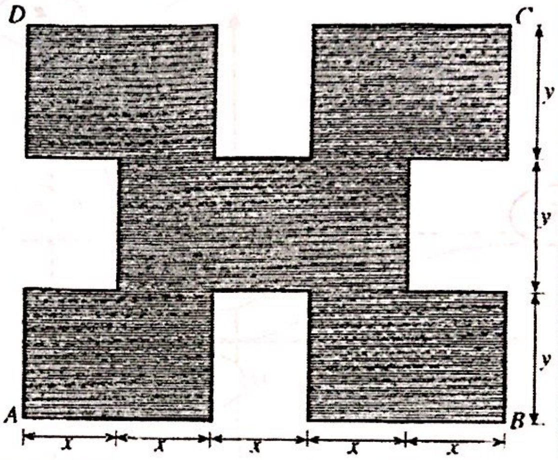
Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x) = \frac{x(4-3x)}{(x-1)^2(x-2)^2}$ for $x \neq 1, 2$.

Sketch the graph of $y = f(x)$ indicating the asymptotes and the turning points.

Using the graph, solve the inequality $\frac{x^2}{(x-1)(x-2)} \leq 0$.

(b) The shaded region shown in the adjoining figure is of area 385 m^2. This region is obtained by removing four identical rectangles each of length y metres and width x metres from a rectangle $ABCD$ of length $5x$ metres and width $3y$ metres. Show that $y = \frac{35}{x}$ and that the perimeter P of the shaded region, measured in metres, is given by $P = 14x + \frac{350}{x}$ for $x > 0$.

Find the value of x such that P is minimum.



(a) $f(x) = \frac{x^2}{(x-1)(x-2)}$ for $x \neq 1, 2$.

Then $f'(x) = \frac{(x-1)(x-2)2x - x^2(2x-3)}{(x-1)^2(x-2)^2}$ (10)

$= \frac{-6x^2 + 4x + 3x^2}{(x-1)^2(x-2)^2}$ (5)

$= \frac{x(4-3x)}{(x-1)^2(x-2)^2}$ for $x \neq 1, 2$. (5)

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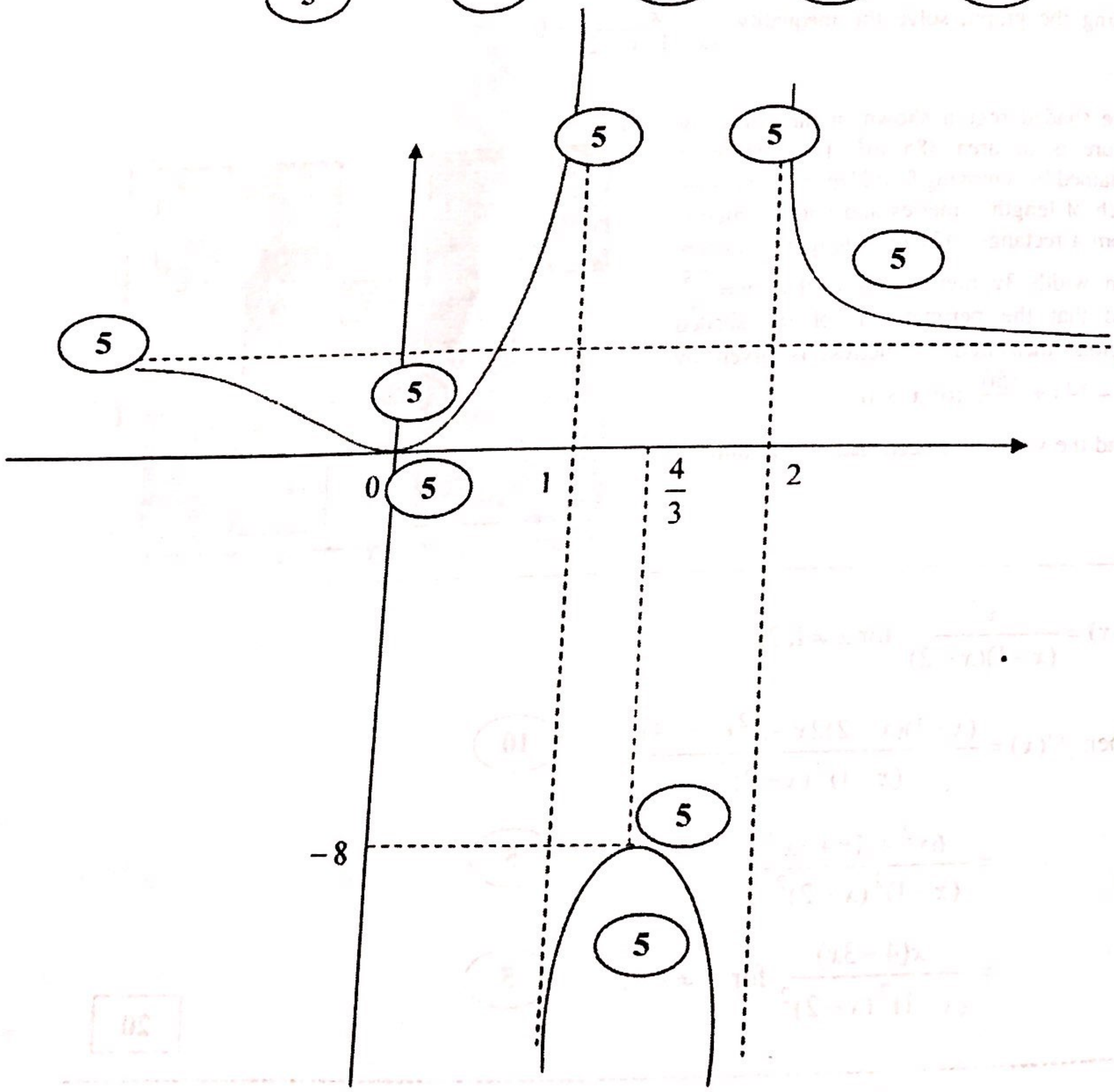
Horizontal Asymptote: $\lim_{x\to\pm\infty} f(x)=1$. Hence, it is $y=1$.

Note that $\lim_{x\to 1-} f(x)=\infty$ and $\lim_{x\to 1+} f(x)=-\infty$ $\lim_{x\to 2-} f(x)=-\infty$ and $\lim_{x\to 2+} f(x)=\infty$.

Vertical Asymptotes: $x=1,2$

$f'(x)=0 \Leftrightarrow x=0$ or $x=\frac{4}{3}$. (5)

	$-\infty<x<0$	$0<x<1$	$1<x<\frac{4}{3}$	$\frac{4}{3}<x<2$	$2<x<\infty$
Sign of $f'(x)$	(-)	(+)	(+)	(-)	(-)
	(5)	(5)	(5)	(5)	(5)



There are two turning points: $(0,0)$ - local minimum and $\left(\frac{4}{3}, -8\right)$ is a local maximum

(5) $x = 0$ or $2 < x < 2$ (5) **70**

(b) Area : $(5x)(3y) - 4xy = 385$ (5)

$11xy = 385$

$xy = 35$

$y = \frac{35}{x}$. (5)

Perimeter: $P = 2(5x + 3y) + 4x + 4y$ (5)

$= 14x + 10y$

$= 14x + \frac{350}{x}; \quad x > 0.$ (5)

$\frac{dP}{dx} = 14 - \frac{350}{x^2}$ (5)

$\frac{dP}{dx} = 0 \Leftrightarrow x^2 = \frac{350}{14} = 25$

(5)

$\therefore x = 5$ (5)

$\frac{dP}{dx} < 0$ for $0 < x < 5$ and $\frac{dP}{dx} > 0$ for $5 < x$

(5) (5)

$\therefore P$ is minimum when $x = 5$. (5) **50**

15.(*a*) (i) Express $\dfrac{1}{x(x+1)^2}$ in partial fractions and **hence**, find $\displaystyle\int \frac{1}{x(x+1)^2}\,dx$.

(ii) Using integration by parts, find $\displaystyle\int xe^{-x}\,dx$ and **hence**, find the area of the region enclosed by the curve $y = xe^{-x}$ and the straight lines $x = 1$, $x = 2$ and $y = 0$.

(*b*) Let $c > 0$ and $I = \displaystyle\int_0^c \frac{\ln(c+x)}{c^2+x^2}\,dx$. Using the substitution $x = c\tan\theta$,

show that $I = \dfrac{\pi}{4c}\ln c + \dfrac{1}{c}J$, where $J = \displaystyle\int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,d\theta$.

Using the formula $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, where a is a constant, show that $J = \dfrac{\pi}{8}\ln 2$.

Deduce that $I = \dfrac{\pi}{8c}\ln(2c^2)$.

(i) $\dfrac{1}{x(x+1)^2} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$ (10)

$1 = A(x+1)^2 + Bx(x+1) + Cx$

$1 = (A+B)x^2 + (2A+B+C)x + A$

By comparing coefficients,

$x^\circ : 1 = A$

$x^1 : 0 = 2A+B+C$ (10)

$x^2 : 0 = A+B$

$\therefore$ $A = 1, B = -1$ and $C = -1$. (10)

$\displaystyle\int \frac{1}{x(x+1)^2}\,dx = \int \frac{1}{x}\,dx - \int \frac{1}{x+1}\,dx - \int \frac{1}{(x+1)^2}\,dx$ (5)

(15) $= \ln|x| - \ln|x+1| + \dfrac{1}{x+1} + C'$, where C' is an arbitrary constant.

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(ii) $\int xe^{-x}dx = -xe^{-x} + \int e^{-x}dx$ (10)

$= -xe^{-x} - e^{-x} + C''$, where C'' is an arbitrary constant. (5)

(5)

Required area $= \int_1^2 xe^{-x}dx$ (5)

$= -(x+1)e^{-x}\Big|_1^2$ (5)

$= 2e^{-1} - 3e^{-2}$. (5)

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(b) Let $x = c\tan\theta$.

Then $dx = c\sec^2\theta\, d\theta$.

When $x = 0$, $\theta = 0$ and when $x = c, \theta = \frac{\pi}{4}$.

(5)

Thus, $I = \int_0^{\frac{\pi}{4}} \frac{\ln c(1+\tan\theta)}{c^2 + c^2\tan^2\theta} \cdot c\sec^2\theta\, d\theta$ (5) (5)

$= \int_0^{\frac{\pi}{4}} \frac{\ln c(1+\tan\theta)}{c^2\sec^2\theta} \cdot c\sec^2\theta\, d\theta$ (5)

$= \frac{1}{c}\int_0^{\frac{\pi}{4}} \{\ln c + \ln(1+\tan\theta)\}d\theta$ (5)

$= \frac{1}{c}\ln c\int_0^{\frac{\pi}{4}} d\theta + \frac{1}{c}\int_0^{\frac{\pi}{4}} \ln\{1+\tan\theta\}d\theta$

$$= \frac{1}{c} \ln c \cdot \theta \Big|_{0}^{\pi/4} + \frac{1}{c} J \quad (5)$$

$$= \frac{\pi}{4c} \ln c + \frac{1}{c} J. \quad (5)$$

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$$J = \int_0^{\frac{\pi}{4}} \ln\left(1 + \tan\left(\frac{\pi}{4} - \theta\right)\right) d\theta \quad (5)$$

$$= \int_0^{\frac{\pi}{4}} \ln\left\{1 + \frac{1 - \tan\theta}{1 + \tan\theta}\right\} d\theta \quad (5)$$

$$= \int_0^{\frac{\pi}{4}} \ln \frac{2}{(1 + \tan\theta)} d\theta$$

$$= \int_0^{\frac{\pi}{4}} \{\ln 2 - \ln(1 + \tan\theta)\} d\theta \quad (5)$$

$$= \ln 2 \cdot \frac{\pi}{4} - J$$

$$\therefore J = \frac{\pi}{8} \ln 2. \quad (5)$$

$$\therefore I = \frac{\pi}{4c} \ln c + \frac{1}{c} \frac{\pi}{8} \ln 2 \quad (5)$$

$$= \frac{\pi}{8c} \{2 \ln c + \ln 2\}$$

$$= \frac{\pi}{8c} \ln(2c^2). \quad (5)$$

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16. Let $m \in \mathbb{R}$. Show that the point $P \equiv (0, 1)$ does not lie on the straight line l given by $y = mx$.

Show that the coordinates of any point on the straight line through P perpendicular to l can be written in the form $(-mt,\ t+1)$, where t is a parameter.

Hence, show that the coordinates of the point Q, the foot of the perpendicular drawn from P to l, are given by $\left(\frac{m}{1+m^2}, \frac{m^2}{1+m^2}\right)$.

Show that, as m varies, the point Q lies on the circle S given by $x^2 + y^2 - y = 0$, and sketch the locus of Q in the xy-plane.

Also, show that the point $R \equiv \left(\frac{\sqrt{3}}{4}, \frac{1}{4}\right)$ lies on S.

Find the equation of the circle S' whose centre lies on the x-axis, and touches S externally at the point R.

Write down the equation of the circle having the same centre as that of S' and touching S internally.

If the point $(0, 1)$ lies on l, then we must have $1 = m \times 0$. i.e. $1 = 0$, a contradiction. (5)

$\therefore (0, 1)$ does not lie on l. (5)

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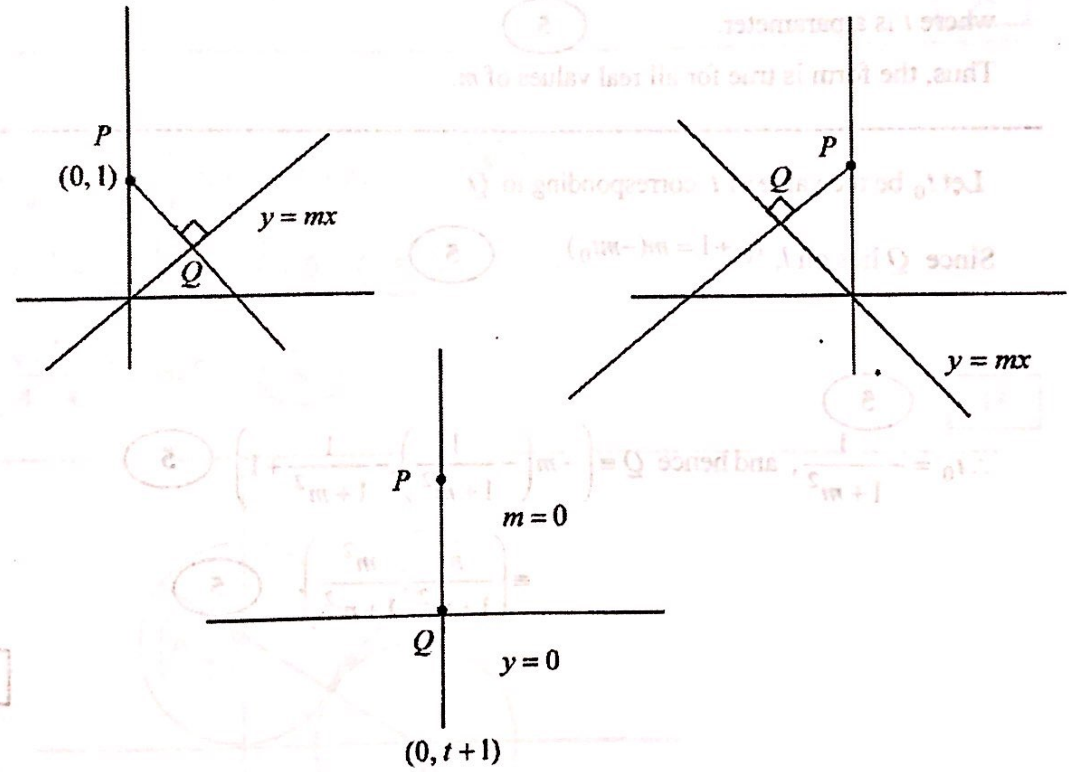

Case(i): $m \neq 0$

In this case, the equation of the line through P perpendicular to l is given by

$y-1=-\frac{1}{m}(x-0)$. (10)

Let us introduce t into this equation by $y-1=-\frac{1}{m}(x-0)=t$ (say). (5)

Then $y=t+1$ (5) and $x=-mt$, (5) where t is a patameter.

Hence, the coordinates of any point on the line through P perpendicular to l can be written in the form $(-mt, t+1)$, where t is a parameter.

Case(ii): $m=0$

In this case, the equation of the line through P perpendicular to l is the y-axis and hence, the coordinates of any point on it can be written in the form $(0, t+1)$, where t is a parameter. (5)

Thus, the form is true for all real values of m.

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Let t_0 be the value of t corresponding to Q.

Since Q lies on l, $t_0+1=m(-mt_0)$. (5)

$\therefore t_0=-\frac{1}{1+m^2}$, (5) and hence $Q\equiv\left(-m\left(-\frac{1}{1+m^2}\right), -\frac{1}{1+m^2}+1\right)$ (5)

$\equiv\left(\frac{m}{1+m^2}, \frac{m^2}{1+m^2}\right)$. (5)

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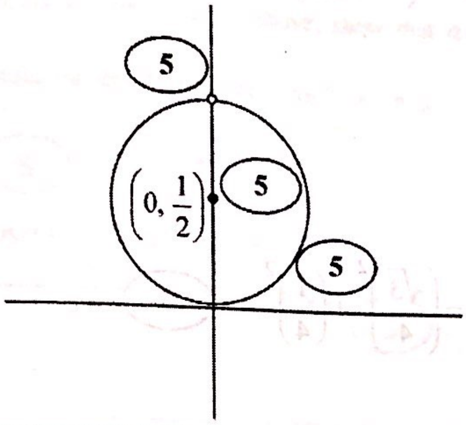
Put $x = \dfrac{m}{1+m^2}$ and $y = \dfrac{m^2}{1+m^2}$ in $x^2 + y^2 - y$: 5

$$x^2 + y^2 - y = \frac{m^2}{(1+m^2)^2} + \frac{m^4}{(1+m^2)^2} - \frac{m^2}{1+m^2} = \frac{m^2(1+m^2)}{(1+m^2)^2} - \frac{m^2}{1+m^2} = 0.$$

5 5

5 Hence Q lies on S.



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Put $x = \dfrac{\sqrt{3}}{4}$ and $y = \dfrac{1}{4}$ in $x^2 + y^2 - y$: 5

$$x^2 + y^2 - y = \frac{3}{16} + \frac{1}{16} - \frac{1}{4} = 0.$$ 5

Hence, $\left(\dfrac{\sqrt{3}}{4}, \dfrac{1}{4}\right)$ lies on S. 5

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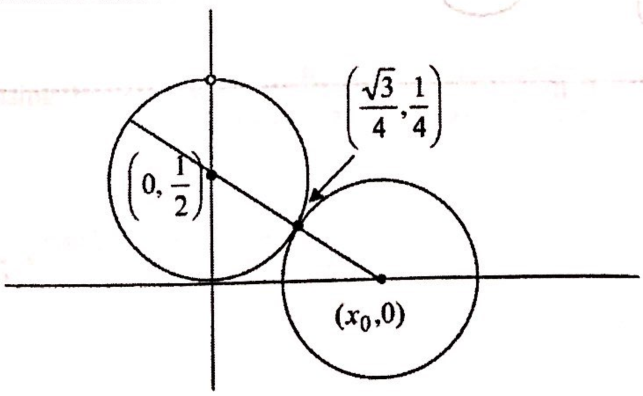

Let x_0 be the x – coordinate of the centre of S'. Then

$$\sqrt{x_0^2+\frac{1}{4}} = \frac{1}{2} + \sqrt{\left(\frac{\sqrt{3}}{4}-x_0\right)^2+\frac{1}{16}}. \quad \textcircled{5}$$

(5) (5)

$$\Rightarrow x_0^2+\frac{1}{4}=\frac{1}{4}+\sqrt{\left(\frac{\sqrt{3}}{4}-x_0\right)^2+\frac{1}{16}}.+\left(\frac{\sqrt{3}}{4}-x_0\right)^2+\frac{1}{16}. \quad \textcircled{5}$$

$$\Rightarrow x_0=\frac{\sqrt{3}}{2}.$$

(5)

Hence the equation of S' is $\left(x-\frac{\sqrt{3}}{2}\right)^2+y^2=\left(\frac{\sqrt{3}}{4}\right)^2+\left(\frac{1}{4}\right)^2$. (5)

i.e. $\left(x-\frac{\sqrt{3}}{2}\right)^2+y^2=\left(\frac{1}{2}\right)^2$.

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The equation of the required circle touching S internally is

$\left(x-\frac{\sqrt{3}}{2}\right)^2+y^2=\left(\frac{3}{2}\right)^2$. (10)

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17\. (a) (i) Show that $\dfrac{2\cos(60^\circ-\theta)-\cos\theta}{\sin\theta}=\sqrt{3}$ for $0^\circ<\theta<90^\circ$.

(ii) In the quadrilateral $ABCD$ shown in the figure, $AB = AD$, $A\hat{B}C = 80^\circ$, $C\hat{A}D = 20^\circ$ and $B\hat{A}C = 60^\circ$.

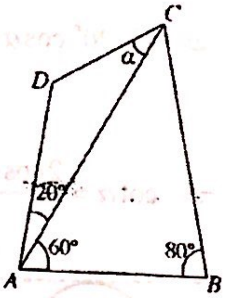
Let $A\hat{C}D = \alpha$. Using the Sine Rule for the triangle ABC, show that $\dfrac{AC}{AB} = 2\cos 40^\circ$.

Next, using the Sine Rule for triangle ADC, show that $\dfrac{AC}{AD}=\dfrac{\sin(20^\circ+\alpha)}{\sin\alpha}$.

Deduce that $\sin(20^\circ+\alpha) = 2\cos 40^\circ \sin\alpha$.

Hence, show that $\cot\alpha = \dfrac{2\cos 40^\circ - \cos 20^\circ}{\sin 20^\circ}$.

Now, using the result in (i) above, show that $\alpha = 30^\circ$.



(b) Solve the equation $\cos 4x + \sin 4x = \cos 2x + \sin 2x$.

(a) (i) $\dfrac{2\left\{\dfrac{1}{2}\cos\theta+\dfrac{\sqrt{3}}{2}\sin\theta\right\}-\cos\theta}{\sin\theta}=\sqrt{3}$. (5) (5) (5)

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(ii) Using the Sine Rule: $\dfrac{AC}{\sin 80^\circ}=\dfrac{AB}{\sin 40^\circ}$. (10)

$\Rightarrow \dfrac{AC}{AB}=\dfrac{2\sin 40^\circ\cos 40^\circ}{\sin 40^\circ}=2\cos 40^\circ$ (5) (5) (5)

Again, using the Sine Rule: $\dfrac{AC}{\sin(\alpha+20^\circ)}=\dfrac{AD}{\sin\alpha}$. (10)

$\Rightarrow \dfrac{AC}{AD}=\dfrac{\sin(20^\circ+\alpha)}{\sin\alpha}$ (5)

Hence, $AB = AD \Rightarrow \dfrac{\sin(20^\circ + \alpha)}{\sin\alpha} = 2\cos 40^\circ.$ (5)

(5)

$\therefore \quad \sin(20^\circ + \alpha) = 2\sin\alpha\cos 40^\circ$

$\Rightarrow \quad \sin 20^\circ \cos\alpha + \cos 20^\circ \sin\alpha = 2\sin\alpha\cos 40^\circ$ (5)

$\Rightarrow \quad \cot\alpha = \dfrac{2\cos 40^\circ - \cos 20^\circ}{\sin 20^\circ}$ (5) **60**

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(10)

(i) with $\theta = 20^\circ \Rightarrow \quad \dfrac{2\cos 40^\circ - \cos 20^\circ}{\sin 20^\circ} = \sqrt{3}$ (5)

$\therefore \cot\alpha = \sqrt{3}$ (5)

(5)

$\Rightarrow \alpha = 30^\circ$. (Since $0^\circ < \alpha < 90^\circ$) **25**

(b) $\cos 4x + \sin 4x = \cos 2x + \sin 2x$

$\Leftrightarrow \sin 4x - \sin 2x = \cos 2x - \cos 4x$ (5)

$\Leftrightarrow 2\cos 3x \sin x = 2\sin 3x \sin x$

(5) (5)

$\Leftrightarrow 2\sin x(\cos 3x - \sin 3x) = 0$ (5)

(5)

$\Leftrightarrow \sin x = 0 \quad \text{or} \quad \cos 3x = \sin 3x$ (5)

$\Leftrightarrow \sin x = 0 \quad \text{or} \quad \tan 3x = 1$ (5) (5) $(\because \cos 3x \neq 0)$

$\Leftrightarrow x = n\pi$ for $n \in \mathbb{Z}$ or $3x = m\pi + \frac{\pi}{4}$ for $m \in \mathbb{Z}$ (5)

$\Leftrightarrow x = n\pi$ for $n \in \mathbb{Z}$ or $x = \frac{m\pi}{3} + \frac{\pi}{12}$ for $m \in \mathbb{Z}$ (5)

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