## G. C. E. (Advanced Level) Examination, August 2011
## Combined Mathematics I
## Scheme of Marking
### Part A

1. Using the **Principle of Mathematical Induction**, prove that $n^3+5n$ is divisible by 3 for every $n \in \mathbb{Z}^+$.

Let $P(n) = n^3 + 5n$.

When $n = 1$, $P(1) = 1 + 5 = 6$ which is divisible by 3. (05)

Thus, the result is true for $n = 1$.

Assume that the result is true for $n = k$.

*i.e.*, assume that $P(k) = k^3 + 5k$ is divisible by 3. (05)

Let $n = k + 1$.

Then $P(k+1) = (k+1)^3 + 5(k+1)$

$= k^3 + 3k^2 + 3k + 1 + 5k + 5$ (05)

$= k^3 + 5k + 3(k^2 + 1k + 2)$ is divisible by 3 since $k^3 + 5k$ is divisible by 3.

Thus, the result is true for $n = k + 1$. (05)

By the **Principle of Mathematical Induction**, $n^3 + 5n$ is divisible by 3 for every $n \in Z^+$.

(05) [25]

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2. Find how many numbers between 2000 and 4000 can be formed using the digits 1, 2, 3 and 4, if repetitions of the digits are (i) not allowed, (ii) allowed. 05

(i) There are 2 ways of finding the first digit and $3 \times 2 \times 1 = 6$ ways of selecting the other three digits. (05)

Thus, there are $2 \times 6 = 12$ ways of selecting a number between 2000 and 4000 using the digits 1, 2, 3 and 4, without the repetition of the digits. (05) [10]

(ii) There are 2 ways of finding the first digit and $4 \times 4 \times 4 = 64$ ways of selecting the other three digits. (~~10~~) 5

Thus, there are $2 \times 64 = 128$ ways of selecting a number between 2000 and 4000 using the digits 1, 2, 3 and 4, with the repetition of the digits.. (05) [15]

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3. Using the binomial expansion for a positive integral index, show that $(1+\sqrt{3})^6 + (1-\sqrt{3})^6 = 416$

Hence, find the integer part of $(1+\sqrt{3})^6$.

$(1+\sqrt{3})^6 = {}^6C_0 + {}^6C_1\sqrt{3} + {}^6C_2(\sqrt{3})^2 + {}^6C_3(\sqrt{3})^3 + {}^6C_4(\sqrt{3})^4 + {}^6C_5(\sqrt{3})^5 + {}^6C_6(\sqrt{3})^6$ (05)

$(1-\sqrt{3})^6 = {}^6C_0 - {}^6C_1\sqrt{3} + {}^6C_2(\sqrt{3})^2 - {}^6C_3(\sqrt{3})^3 + {}^6C_4(\sqrt{3})^4 - {}^6C_5(\sqrt{3})^5 + {}^6C_6(\sqrt{3})^6$ (05)

$(1+\sqrt{3})^6 + (1-\sqrt{3})^6 = 2\{{}^6C_0 + {}^6C_2(\sqrt{3})^2 + {}^6C_4(\sqrt{3})^4 + {}^6C_6(\sqrt{3})^6\}$

$= 2\{1 + 15 \times 3 + 15 \times 9 + 27\}$ (05)

$= 2\{1 + 180 + 27\} = 416$ (05) [20]

$(1+\sqrt{3})^6 = 416 - (1-\sqrt{3})^6$

$0 < (1-\sqrt{3})^6 < 1 \quad \because 0 < \sqrt{3} - 1 < 1$

Thus, the integer part of $(1+\sqrt{3})^6$ is 415. (05) [05]

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4. Show that $\lim_{x \to 0} \dfrac{\sqrt{4+3\sin x} - \sqrt{4-3\sin x}}{2x} = \dfrac{3}{4}$.

$\lim_{x \to 0} \dfrac{\sqrt{4+3\sin x} - \sqrt{4-3\sin x}}{2x}$

$= \lim_{x \to 0} \dfrac{\sqrt{4+3\sin x} - \sqrt{4-3\sin x}}{2x} \times \dfrac{\sqrt{4+3\sin x} + \sqrt{4-3\sin x}}{\sqrt{4+3\sin x} + \sqrt{4-3\sin x}}$ (05)

$= \lim_{x \to 0} \dfrac{4+3\sin x - (4-3\sin x)}{2x} \times \dfrac{1}{\sqrt{4+3\sin x} + \sqrt{4-3\sin x}}$ (05)

$= 3\lim_{x \to 0} \dfrac{\sin x}{x} \times \lim_{x \to 0} \dfrac{1}{\sqrt{4+3\sin x} + \sqrt{4-3\sin x}} = (3 \times 1) \times \left(\dfrac{1}{\sqrt{4}+\sqrt{4}}\right) = \dfrac{3}{4}$

(05) (05) (05) [25]

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5. Find constants $A$ and $B$ such that $\dfrac{d}{dx}\{e^{2x}(A\sin 3x + B\cos 3x)\} = 13e^{2x}\sin 3x$.

Hence, find $\int e^{2x}\sin 3x\,dx$.

$\dfrac{d}{dx}\{e^{2x}(A\sin 3x + B\cos 3x)\}$

05

$= e^{2x}(3A\cos 3x - 3B\sin 3x) + 2e^{2x}(A\sin 3x + B\cos 3x)$ (05)

$= e^{2x}\{(3A + 2B)\cos 3x + (2A - 3B)\sin 3x\}$

$= 13e^{2x}\sin 3x$ if and only if $3A + 2B = 0$ and $2A - 3B = 13 \Leftrightarrow \underline{A = 2 \text{ and } B = -3}$

(05) (~~05~~) [15]

Hence, $\int e^{2x}\sin 3x\,dx = e^{2x}\left(\dfrac{2}{13}\sin 3x - \dfrac{3}{13}\cos 3x\right) + C$, where $C$ is an arbitrary constant.

(05) (05) [10]

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6. Find the equation of the straight line parallel to the straight line $3y + 2x + 5 = 0$, and passing through the point that divides the straight line joining the points (2, 3) and (−1, 2) externally in the ratio 3 : 2.

Since the required straight line parallel to the straight line $3y + 2x + 5 = 0$, the equation of the required straight line can be written as $3y + 2x + c = 0$, where the constant $c$ to be determined.

(05)

Let $(x_0, y_0)$ be the coordinates of the point that divides the straight line joining the points (2, 3) and (-1, 2) externally in the ratio 3 : 2.

Then, $x_0 = \dfrac{2 \times 2 + (-3) \times (-1)}{-3+2} = -7$ and $y_0 = \dfrac{2 \times (3) + (-3) \times (2)}{-3+2} = 0$.

(05) (05)

Since the point $(-7, 0)$ lies on the line we have $3(0) + 2(-7) + c = 0$ which implies $c = 14$.

(05)

Thus, the equation of the required straight line is $3y + 2x + 14 = 0$. (05) [25]

7. A curve is given by $x = 3t$, $y = \frac{3}{t}$ where $t$ is a non-zero parameter. Show that the equation of the tangent to the curve at the point $\left(3t, \frac{3}{t}\right)$ is $x + t^2 y = 6t$.

Deduce that, as $t$ varies, the area of the triangular region bounded by the coordinate axes and this tangent is a constant.

05

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = -\frac{3}{t^2} \cdot \frac{1}{3} = -\frac{1}{t^2}$. (05)

The equation of the tangent is $\frac{\left(y - \frac{3}{t}\right)}{(x - 3t)} = -\frac{1}{t^2}$. (05)

i.e $x + t^2 y = 6t$. (05) [15]

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This tangent cuts the $x$-axis at the point $A = \left(0, \frac{6}{t}\right)$ and the $y$-axis at the point $B = (6t, 0)$.

(05)

The area of the triangle $OAB$ is $\frac{1}{2} \times 6t \times \left(\frac{6}{t}\right) = 18$ sq. unit which is independent of $t$.

(05) [10]

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8. Find the equations of the two circles, each of radius $\sqrt{2}$, touching the straight line $x + y + 1 = 0$ and having the centres on the $y$-axis.

Centre of each circle is of the form $(0, b)$. (05)

The perpendicular distance from $(0, b)$ to the line $x + y + 1 = 0$ is $\left|\frac{b+1}{\sqrt{2}}\right| = \sqrt{2}$. (05)

$\Rightarrow (b+1)^2 = 4 \Rightarrow b^2 + 2b - 3 = 0 \Rightarrow b = 1$ and $-3$. (05)

Thus, the equations of the two circles are $x^2 + (y-1)^2 = 2$ and $x^2 + (y+3)^2 = 2$.

(05) (05) [25]

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9. The length of the tangent from a point $P$ to the circle $x^2 + y^2 - 12x = 0$ is twice the length of the tangent from the point $P$ to the circle $x^2 + y^2 - 9 = 0$. Show that the point $P$ lies on the circle $x^2 + y^2 + 4x - 12 = 0$.

Let $P = (\alpha, \beta)$

The length of the tangent from a point $P$ to the circle $x^2 - 12x + y^2 = 0$ is $\sqrt{\alpha^2 - 12\alpha + \beta^2}$.

(05)

The length of the tangent from a point $P$ to the circle $x^2 + y^2 - 9 = 0$ is $\sqrt{\alpha^2 + \beta^2 - 9}$. (05)

$\alpha^2 - 12\alpha + \beta^2 = 4(\alpha^2 + \beta^2 - 9)$. (05)

i.e. $3\alpha^2 + 3\beta^2 + 12\alpha - 36 = 0$.

i.e. $\alpha^2 + \beta^2 + 4\alpha - 12 = 0$. (05)

Thus, the point $P$ lies on the circle $x^2 + y^2 + 4x - 12 = 0$. (05) [25]

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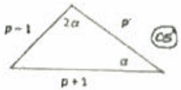
10. The sides of a triangle are $p - 1$, $p$ and $p + 1$, where $p$ is a real number such that $p > 1$. If the largest angle of the triangle is twice the smallest angle of the triangle, using the **Sine rule** and the **Cosine rule** find the value of $p$.

Using Sine rule we have

$\frac{\sin\alpha}{p-1} = \frac{\sin 2\alpha}{p+1} \Rightarrow \cos\alpha = \frac{p+1}{2(p-1)}$

(05) (05)




(05)

Sin rule (10)

Using Cosine rule we have

$\cos\alpha = \frac{(p+1)^2 + p^2 - (p-1)^2}{2p(p+1)} = \frac{p+4}{2(p+1)}$

(05) (05)

Thus, we have $\frac{p+1}{2(p-1)} = \frac{p+4}{2(p+1)} \Rightarrow p^2 + 2p + 1 = p^2 + 3p - 4 \Rightarrow p = 5$. (05) [25]

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11.(a) Let $\alpha$ and $\beta$ be the roots of the quadratic equation $ax^2+bx+c=0$, where $a$, $b$ and $c$ are real numbers. Show that $\alpha$ and $\beta$ are both

(i) real, if and only if $b^2-4ac \geq 0$,

(ii) purely imaginary, if and only if $b=0$ and $ac>0$.

Find the quadratic equation whose roots are $\alpha^2$ and $\beta^2$.

Show that the roots of this quadratic equation are both real, if and only if **either** $\alpha$ and $\beta$ are both real **or** $\alpha$ and $\beta$ are both purely imaginary.

(b) Let $f(x)=x^3-3abx-(a^3+b^3)$, where $a$ and $b$ are real numbers. Show that $(x-a-b)$ is a factor of $f(x)$. Find the other factor of $f(x)$ in quadratic form.

Hence or otherwise, show that if $a$ and $b$ are distinct, then $f(x)=0$ has only one real root

**Deduce** that $x^3-9x-12=0$ has only one real root and find it.

(a) (i) $ax^2+bx+c=a\left\{x^2+\frac{b}{a}x+\frac{c}{a}\right\}$

$$=a\left\{\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}\right\}$$

$$=a\left\{\left(x+\frac{b}{2a}\right)^2-\left(\frac{b^2-4ac}{4a^2}\right)\right\}$$

$ax^2+bx+c=0$ if and only if $\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2} \rightarrow (1)$ ~~(05)~~ 10

$x$ has two real values satisfying equation (1) if and only if $b^2-4ac\geq 0$ (05)

~~i.e. $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ (both real) if and only if $b^2-4ac\geq 0$ (05)~~

i.e. $ax^2+bx+c=0$ has real roots if and only if $b^2-4ac\geq 0$ (05) [20]

(ii) $x$ has two complex values satisfying equation (1) if and only if $b^2-4ac<0$ (05)

i.e. $x=\frac{-b\pm i\sqrt{\cancel{b^2-4ac}\,4ac-b^2}}{2a}$ (both complex) if and only if $b^2-4ac<0$ 05

$x=\frac{\sqrt{\cancel{b^2-4ac}\,4ac-b^2}}{2a}i$ (both pure imaginary) if and only if $b^2-4ac<0$ and $b=0$ (05)

i.e. $x=\frac{\sqrt{\cancel{b^2-4ac}\,4ac-b^2}}{2a}i$ (both pure imaginary) if and only if $ac>0$ and $b=0$ ~~(05)~~

i.e. $ax^2+bx+c=0$ has pure imaginary roots if and only if $ac>0$ and $b=0$ (05) [20]

**Aliter for (a) (i):**

Since $\alpha$ and $\beta$ are the roots of the quadratic equation $ax^2+bx+c=0$ we have

$\alpha+\beta=-\frac{b}{a}\rightarrow(1)$ and $\alpha\beta=\frac{c}{a}$. (05)

$(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta=\left(-\frac{b}{c}\right)^2-4\left(\frac{c}{a}\right)=\frac{b^2-4ac}{a^2}$ (05)



$\alpha-\beta=\pm\frac{\sqrt{b^2-4ac}}{a}\rightarrow(2)$ if and only if $b^2-4ac\geq 0$. (05)

(1) and (2) imply that the two roots of the quadratic equation $ax^2+bx+c=0$ are $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$, and both are real. (05)

Both roots are real if and only if $b^2-4ac\geq 0$. [20]

**Aliter for (a) (ii):**

$\alpha-\beta=\pm\frac{\sqrt{\cancel{b^2-4ac}\,4ac-b^2}}{a}i\rightarrow(3)$ if and only if $b^2-4ac<0$ (05)

(1) and (3) imply that the two roots of the quadratic equation $ax^2+bx+c=0$ are $\frac{-b+i\sqrt{\cancel{b^2-4ac}\,4ac-b^2}}{2a}$ and $\frac{-b-i\sqrt{\cancel{b^2-4ac}\,4ac-b^2}}{2a}$, and they are complex. (05)

If $b=0$, the real part of each of them are zero and hence both of them are pure imaginary. (05)

i.e., both roots are pure imaginary if and only if $ac>0$ and $b=0$. (05) [20]

The quadratic equation whose roots are $\alpha^2$ and $\beta^2$, can be written as $x^2-(\alpha^2+\beta^2)x+\alpha^2\beta^2=0$ (05)

i.e. $x^2-\left\{(\alpha+\beta)^2-2\alpha\beta\right\}x+\alpha^2\beta^2=0$

i.e., $x^2-\left\{\left(-\frac{b}{a}\right)^2-2\left(\frac{c}{a}\right)\right\}x+\left(\frac{c}{a}\right)^2=0$ (05)

i.e., $a^2x^2-(b^2-2ac)x+c^2=0$. (05) [15]

The roots of this quadratic equation are both real if and only if $\left[-(b^2-2ac)\right]^2-4a^2c^2\geq 0$. (05)

i.e., the roots are both real if and only if $\left\{(b^2-2ac)-2ac\right\}\left\{(b^2-2ac)+2ac\right\}\geq 0$.

i.e., the roots are both real if and only if $b^2(b^2-4ac)\geq 0$. (05)

$b^2(b^2-4ac)\geq 0$ if and only if **either** $b^2-4ac\geq 0$ and $b$ is any real value **or** $b^2-4ac<0$ and $b=0$. (10) / 0

i.e., $b^2(b^2-4ac)\geq 0$ if and only if **either** $\alpha$ and $\beta$ are both real **or** $\alpha$ and $\beta$ are both purely imaginary. (05)

Hence, we have the result. [25]

**Aliter:**

$\alpha$ and $\beta$ are the roots of the quadratic equation $ax^2+bx+c=0$.

Thus, $\alpha$ and $\beta$ are both

(i) real if and only if $b^2-4ac\geq 0$, (05)

(ii) purely imaginary if and only if $ac>0$ and $b=0$, (05)

(iii) complex if and only if $b^2-4ac<0$ and $b\neq 0$. (05)

$\alpha^2$ and $\beta^2$ are both cannot be real if $\alpha$ and $\beta$ are both complex. (05)

Thus, $\alpha^2$ and $\beta^2$ are both real if and only if $\alpha$ and $\beta$ are both real or both purely imaginary. (05) [25]

(b) $f(a+b) = (a+b)^3 - 3ab(a+b) - (a^3+b^3) = 0$. (05)

Thus, $(x-a-b)$ is a factor of $f(x)$. [05]

$f(x) = x^3 - 3abx - (a^3+b^3) = (x-a-b)(x^2+(a+b)x+a^2-ab+b^2)$

(05) (05) (05)

$x^2$ coeff. x coeff. constant [15]

$f(x) = (x-a-b)\{x^2+(a+b)x-3ab+(a+b)^2\}$

If $f(x)=0$ has only one real root then $x^2+(a+b)x-3ab+(a+b)^2=0$ should have complex roots. (05)

For $x^2+(a+b)x-3ab+(a+b)^2=0$ to have complex roots we must have

$(a+b)^2 - 4\{(a+b)^2-3ab\} < 0$ (05)

i.e., we must have $(a+b)^2 - 4ab > 0$.

i.e., we must have $(a-b)^2 > 0$. (05)

i.e., $a$ and $b$ should be distinct. (05) [20]

If $ab=3$ and $a^3+b^3=12$ then $x^3-9x-12 = x^3-3abx-(a^3+b^3)$.

i.e., if $a^3+\left(\frac{3}{a}\right)^3 = 12$ then $x^3-9x-12 = x^3-3abx-(a^3+b^3)$.

i.e., if $a^6-12a^3+27=0$ then $x^3-9x-12 = x^3-3abx-(a^3+b^3)$.

i.e., if $(a^3-9)(a^3-3)=0$ then $x^3-9x-12 = x^3-3abx-(a^3+b^3)$.

i.e., if $(a^3-9)(a^3-3)=0$ then $x^3-9x-12 = x^3-3abx-(a^3+b^3)$.

i.e., if $a=3^{\frac{2}{3}}$ (05) and $b=3^{\frac{1}{3}}$ (05) then $x^3-9x-12 = x^3-3abx-(a^3+b^3)$. (05)

If $a=3^{\frac{2}{3}}$ and $b=3^{\frac{1}{3}}$ then $x^3-9x-12=0$ can be written as $x^3-3abx-(a^3+b^3)=0$.

Since $a=3^{\frac{2}{3}}$ and $b=3^{\frac{1}{3}}$ are distinct (05) by above result $x^3-9x-12=0$ has only one real root. (05) [25]

This real root is $3^{\frac{2}{3}}+3^{\frac{1}{3}}$. (05) [05]

$\Delta = (a+b)^2 - 4\{(a+b)^2 - 3ab\}$

$= -3\{(a+b)^2 - 4ab\}$

$= -3(a-b)^2$.

$a \neq b$ and $\Delta < 0$ $f(x)=0$ has only one real root

a & b are distinct



12. (a) Let $u_r = \frac{1}{(2r-1)(2r+1)(2r+3)}$ for $r \in \mathbb{Z}^+$.

Find $\frac{u_{r+1}}{u_r}$ in terms of $r$.

Hence, show that $(2r-1)u_r - (2r+1)u_{r+1} = 4u_{r+1}$ for $r = 1, 2, 3, \ldots$.

Deduce that $\sum_{r=1}^{n} u_r = \frac{1}{12} - \frac{1}{4(2n+1)(2n+3)}$.

Is the series $\sum_{r=1}^{\infty} u_r$ convergent? Justify your answer.

(b) Draw the graph of $y = |2x-8|$.

Hence, draw the graph of $y = -|2x-8|$.

Draw the graphs of $y = 4-|2x-8|$, and $y = |2x-10|$ in the same figure.

Hence or otherwise, find the set of real values of $x$ satisfying the inequality $|2x-10|+|2x-8| \le 4$.

(a) $u_r = \frac{1}{(2r-1)(2r+1)(2r+3)} \Rightarrow u_{r+1} = \frac{1}{(2r+1)(2r+3)(2r+5)}$ 05

$\frac{u_{r+1}}{u_r} = \frac{(2r-1)(2r+1)(2r+3)}{(2r+1)(2r+3)(2r+5)} = \frac{(2r-1)}{(2r+5)}$ (05) 10 [05]

i.e. $(2r-1)u_r = (2r+5)u_{r+1} = (2r+1)u_{r+1} + 4u_{r+1}$ (05)

i.e. $(2r-1)u_r - (2r+1)u_{r+1} = 4u_{r+1}$ for $r = 1, 2, 3, \ldots$ [05]

$4u_1 = 5u_1 - u_1$ (05)

When $r=1$, $4u_2 = u_1 - 3u_2$ (05)

When $r=2$, $4u_3 = 3u_2 - 5u_3$ (05)

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When $r=n-2$ $4u_{n-1} = (2n-5)u_{n-2} - (2n-3)u_{n-1}$ (05)

When $r=n-1$ $4u_n = (2n-3)u_{n-1} - (2n-1)u_n$ (05)

$4\sum_{r=1}^{n} u_r = 5u_1 - (2n-1)u_n = \frac{5}{1.3.5} - \frac{(2n-1)}{(2n-1)(2n+1)(2n+3)}$

(05) (05)

$\sum_{r=1}^{n} u_r = \frac{1}{12} - \frac{1}{4(2n+1)(2n+3)}$. (05) [40]

Yes, the series $\sum_{r=1}^{\infty} u_r$ is convergent. (05) [05]

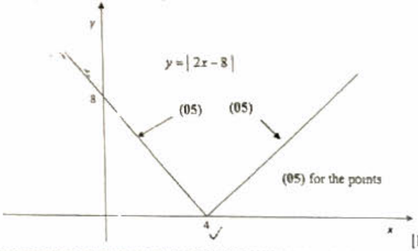
Reason:

$\lim_{n\to\infty}\sum_{r=1}^{n} u_r = \lim_{n\to\infty}\left\{\frac{1}{12} - \frac{1}{4(2n+1)(2n+3)}\right\} = \frac{1}{12} - \lim_{n\to\infty}\frac{1}{4n^2\left(2+\frac{1}{n}\right)\left(2+\frac{3}{n}\right)} = \frac{1}{12}$ (05)

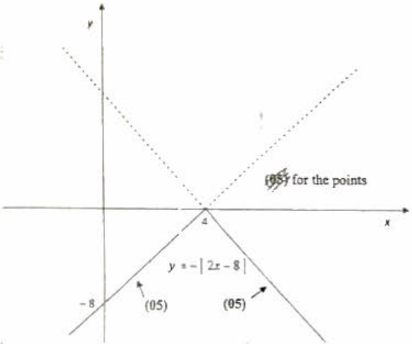
(05)

Since the sum of the first $n$ terms of the series tends to a finite limit as $n$ tends to infinity, the series is convergent (05) [10]

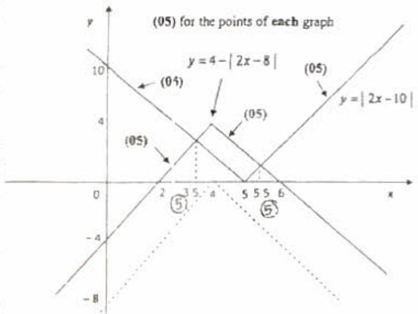
(b)




[15]




[10]




[10] $y=4-|2x-8|$

[15]

35

It is clear from the above graphs that the graph of $y=4-|2x-8|$ is above the graphs of $y=|2x-10|$ when $3.5\le x\le 5.5$ (10)

Thus, $|2x+10|+|2x-8|<4$ hold when $3.5\le x\le 5.5$. (10) [20]



13. (a) Let $\mathbf{A}=\begin{pmatrix}2&1\\-1&3\end{pmatrix}$ and $\lambda,\mu\in\mathbb{R}$. Find the values of $\lambda$ and $\mu$ such that $\mathbf{A}(\lambda\mathbf{A}+\mu\mathbf{I})=\mathbf{I}$, where $\mathbf{I}$ is the $2\times 2$ identity matrix.

Hence, find $\mathbf{A}^{-1}$.

(b) Let $P$, $Q$ and $R$ be three distinct points which represent complex numbers $z_0$, $z_1$ and $z_2$ respectively in the Argand diagram.

If $PQ=PR$ and $\theta$ is the angle measured from $PQ$ to $PR$ in the anti-clockwise sense, show that $z_2-z_0=(z_1-z_0)(\cos\theta+i\sin\theta)$.

The points $A$, $B$, $C$ and $D$, taken in the anti-clockwise sense, form a square in the Argand diagram. Let $1-i$ and $z$ be the complex numbers represented by the points $A$ and $B$ respectively. Find the complex numbers represented by the points $C$ and $D$ in terms of $z$.

If $C$ varies such that $AC=2$, find the locus of $B$ in the Argand diagram.

(a) $\mathbf{A}(\lambda\mathbf{A}+\mu\mathbf{I})=\mathbf{I}\Rightarrow$

$$\begin{pmatrix}2&1\\-1&3\end{pmatrix}\left\{\lambda\begin{pmatrix}2&1\\-1&3\end{pmatrix}+\mu\begin{pmatrix}1&0\\0&1\end{pmatrix}\right\}=\begin{pmatrix}1&0\\0&1\end{pmatrix} \quad (05)$$

$$i.e.,\ \begin{pmatrix}2&1\\-1&3\end{pmatrix}\begin{pmatrix}2\lambda+\mu&\lambda\\-\lambda&3\lambda+\mu\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix} \quad (10)$$

$$i.e.,\ \begin{pmatrix}3\lambda+2\mu&5\lambda+\mu\\-5\lambda-\mu&8\lambda+3\mu\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}\Rightarrow (10)$$

$3\lambda+2\mu=1$, $5\lambda+\mu=0$ and $8\lambda+3\mu=1\Rightarrow$ $\lambda=-\frac{1}{7}$ and $\mu=\frac{5}{7}$.

(05) (05) (05) (05) (05) 45 [50]

$$\mathbf{A}^{-1}=\mathbf{A}^{-1}\mathbf{A}(\lambda\mathbf{A}+\mu\mathbf{I})=\lambda\mathbf{A}+\mu\mathbf{I} \quad (05)$$

$$=\begin{pmatrix}2\lambda+\mu&\lambda\\-\lambda&3\lambda+\mu\end{pmatrix} \quad (05)$$

$$=\begin{pmatrix}-\frac{2}{7}+\frac{5}{7}&-\frac{1}{7}\\\frac{1}{7}&-\frac{3}{7}+\frac{5}{7}\end{pmatrix}=\begin{pmatrix}\frac{3}{7}&-\frac{1}{7}\\\frac{1}{7}&\frac{2}{7}\end{pmatrix}=\frac{1}{7}\begin{pmatrix}3&-1\\1&2\end{pmatrix}.$$

(05) (05) or (05) [20]

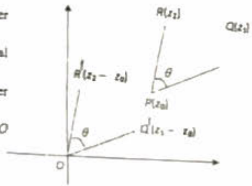
(b) Draw the line segment $OQ'$ through $O$ equal and parallel to $PQ$. (05)

Then the point $Q'$ represents the complex number $z_1-z_0$ in the Argand diagram. (05)

Similarly, draw the line segment $OR'$ through $O$ equal and parallel to $PR$. (05)

Then the point $R'$ represents the complex number $z_2-z_0$ in the Argand diagram. (05)

The point $R'$ can be obtained by rotating $OQ'$ about $O$ through an angle $\theta$ in the anti-clockwise sense. (05)




Thus, the point $R'$ represents the complex number $(z_1-z_0)(\cos\theta+i\sin\theta)$ (05)

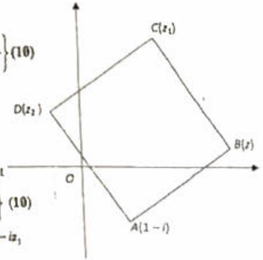
Hence, we have $z_2-z_0=(z_1-z_0)(\cos\theta+i\sin\theta)$. (05) [35]

Consider the three points $A$, $B$ and $D$.
Applying the above result we get

$z_2-(1-i)=\{z-(1-i)\}\left\{\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right\}$ (10)

i.e., $z_2=i\{z-(1-i)\}+1-i$

i.e., $z_2=iz-2i=i(z-2)$ (05)




Consider the three points $C$, $D$ and $B$.
Applying the above result again we get

$z-z_1=\{i(z-2)-z_1\}\left\{\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right\}$ (10)

i.e., $z-z_1=i\{i(z-2)-z_1\}=-(z-2)-iz_1$

i.e., $z_1=\frac{2(z-1)}{1-i}=\frac{2(z-1)(1+i)}{(1-i)(1+i)}=(1+i)(z-1)$. (05)

$AC=|z_1-(1-i)|=|(1+i)(z-1)-(1-i)|=|(1+i)z-2|=|(x-y-2)+i(x+y)|$

(05) (05)

Since $AC=2$ we have $(x-y-2)^2+(x+y)^2=4$ (05)

i.e., $x^2+y^2-4x+4y-4+x^2+y^2=4$

i.e., $x^2+y^2-2x+2y=0$

i.e., $(x-1)^2+(y+1)^2=2$ (05) [50]

---

$z-1=iz-i$

$(z-1)=i(z-1)$

14. (a) Let $f(x)=2x^3+ax^2+bx$ for $x\in\mathbb{R}$, where $a$ and $b$ are real constants. Suppose that $f'(3)=12$ and $f''(3)=18$, where $f'$ and $f''$ have the usual meaning.

Find the values of $a$ and $b$.

For these values of $a$ and $b$, sketch the graph of $y=f(x)$ indicating the turning points.

Hence, find the number of solutions of the equation $2x^2+ax+b=\frac{3}{x}$.

(b) A closed rectangular box with a square base is made of thin cardboard. The volume of the box is 8192 cm$^3$. Let the length of a side of the square base be $4x$ cm. A circular hole of radius $x$ cm is cut out from the top square face. Show that the surface area $A$ cm$^2$ of the box with the hole is given by $A=(32-\pi)x^2+\frac{8192}{x}$.

Hence, show that $A$ is minimum when $x=\frac{16}{\sqrt[3]{32-\pi}}$.

(a) $f(x)=2x^3+ax^2+bx$

$f'(x)=6x^2+2ax+b$ (05) and $f''(x)=12x+2a$ (05)

Since $f'(3)=12$ and $f''(3)=18$, we then have

$54+6a+b=12$ (05) and $36+2a=18$. (05)

Thus, we have $a=-9$ (05) and $b=12$. (05)

[30]

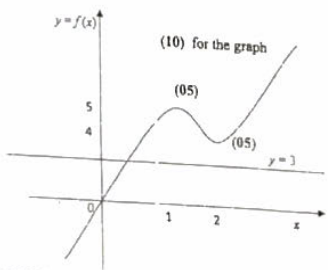
So, we have $f(x)=2x^3-9x^2+12x$ and

$f'(x)=6x^2-18x+12=6(x-1)(x-2)=0\Leftrightarrow x=1$ and $x=2$

~~(05)~~ (05) (05)

| | $x<1$ | $1<x<2$ | $2<x$ |
|---|---|---|---|
| Sign of $f'(x)$ | (+) | (−) | (+) |
| ~~(10)~~ 15 | $f(x)$ is increasing | $f(x)$ is decreasing | $f(x)$ is increasing |

Note that $f(0)=0$, $f(1)=5$ and $f(2)=4$.




[illegible] [illegible] (−5) marks

[45]

$2x^2 + ax + b = \frac{3}{x} \Leftrightarrow f(x) = 3$. (05)

The number of solutions of $2x^2 + ax + b = \frac{3}{x}$ is the same as the number of points of intersection of the curve $y = f(x)$ and the line $y = 3$. (05)

From the above graph we see that there is only one point of intersection and the required answer is 1. (05) [15]

---

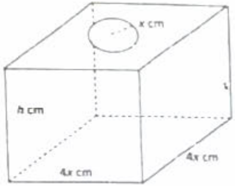
(b) Let the height of the box be $h$ cm.

$A = (2\times 16x^2 + 4\times 4xh) - \pi x^2$ (10)

The volume of the box is $16x^2h$ and is equal to 8192. (10) 5

$16x^2h = 8192 \Rightarrow 16h = \frac{8192}{x^2}$ (05)

$A = (32-\pi)x^2 + \frac{8192}{x}$. (05)




[30]

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$\frac{dA}{dx} = 2(32-\pi)x - \frac{8192}{x^2}$. (05)

$\therefore\ \frac{dA}{dx} = 0 \Leftrightarrow x^3 = \frac{4096}{(32-\pi)} = \frac{(16)^3}{(32-\pi)} \Leftrightarrow x = \frac{16}{\sqrt[3]{(32-\pi)}}$.

(05) (05) (05)

$\frac{dA}{dx} < 0$, when $x < \frac{16}{\sqrt[3]{(32-\pi)}}$ and $\frac{dA}{dx} > 0$, when $x > \frac{16}{\sqrt[3]{(32-\pi)}}$. (05)

Thus, $A$ is minimum, when $x = \frac{16}{\sqrt[3]{(32-\pi)}}$. (05) [30]

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15. (a) Using the method of Integration by Parts, evaluate $\int_1^e x^{\frac{3}{2}} \ln x\, dx$.

(b) Let $t = \tan x$.

Show that $\cos 2x = \frac{1-t^2}{1+t^2}$, $\sin 2x = \frac{2t}{1+t^2}$ and $\frac{dx}{dt} = \frac{1}{1+t^2}$.

Hence, show that $\int_0^{\frac{\pi}{4}} \frac{1}{4\cos 2x + 3\sin 2x + 5}\, dx = \frac{1}{12}$.

(c) Let $a$ and $b$ be distinct real numbers.

Find constants $A$ and $B$ such that $\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$ for $x \in \mathbb{R} - \{a, b\}$.

By replacing $x$, $a$ and $b$ appropriately in the above equation, write down $\frac{1}{(x^2+a^2)(x^2+b^2)}$ in partial fractions and hence, find $\int \frac{1}{(x^2+a^2)(x^2+b^2)}\, dx$.

(a) $\int_1^e x^{\frac{3}{2}} \ln x\, dx = \int_1^e \ln x\, d\left(\frac{2}{5}x^{\frac{5}{2}}\right)$ (10)

$= \left[\frac{2}{5}x^{\frac{5}{2}} \ln x\right]_1^e - \frac{2}{5}\int_1^e x^{\frac{5}{2}}\left(\frac{1}{x}\right) dx$ (10) 5 + 5

$= \frac{2}{5}e^{\frac{5}{2}} - \frac{2}{5}\int_1^e x^{\frac{3}{2}} dx \quad \because \ln e = 1$ and $\ln 1 = 0$ (05)

$= \frac{2}{5}e^{\frac{5}{2}} - \left(\frac{2}{5}\right)^2 \left[x^{\frac{5}{2}}\right]_1^e$ (05)

$= \frac{2}{5}e^{\frac{5}{2}} - \left(\frac{2}{5}\right)^2 e^{\frac{5}{2}} + \left(\frac{2}{5}\right)^2$ (05)

[35]

---

(b) $\cos 2x = \cos^2 x - \sin^2 x = \frac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x} = \frac{1-\tan^2 x}{1+\tan^2 x} = \frac{1-t^2}{1+t^2}$. (05)

$\sin 2x = 2\cos x \sin x = \frac{2\cos x \sin x}{\cos^2 x + \sin^2 x} = \frac{2\tan x}{1+\tan^2 x} = \frac{2t}{1+t^2}$. (05)

$t = \tan x \Rightarrow \frac{dt}{dx} = \sec^2 x = 1 + \tan^2 x = 1 + t^2$

Thus, we have $\frac{dx}{dt} = \frac{1}{1+t^2}$. (05)

[15]

---

Limit 5 + Substitute 5 + Simplify 5

$\int_0^{\frac{\pi}{4}} \frac{1}{4\cos 2x + 3\sin 2x + 5}\, dx = \int_0^1 \frac{1}{4(1-t^2) + 6t + 5(1+t^2)}\, dt$ (15)

$= \int_0^1 \frac{1}{t^2 + 6t + 9}\, dt$

$= \int_0^1 \frac{1}{(t+3)^2}\, dt = \left[-\frac{1}{(t+3)}\right]_0^1 = -\frac{1}{4} + \frac{1}{3} = \frac{1}{12}$

(05) (05) (05)

[30]

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Coverup mtd
$x=a,\ x=b$ substitute
↑
no marks,

(c) Let $A=\dfrac{1}{a-b}$ and $B=\dfrac{1}{b-a}$.
(05) (05)

Then, $\dfrac{A}{x-a}+\dfrac{B}{x-b}=\dfrac{1}{a-b}\left\{\dfrac{1}{x-a}-\dfrac{1}{x-b}\right\}=\dfrac{1}{(x-a)(x-b)}$ for $x\in\Re-\{a,b\}$.
(05)

Thus, $\dfrac{1}{(x-a)(x-b)}=\dfrac{1}{(a-b)}\dfrac{1}{(x-a)}-\dfrac{1}{(a-b)}\dfrac{1}{(x-b)}\rightarrow(1)$ (05)

$\boxed{10/0}$ [20]

---

Replacing $x$ by $x^2$, $a$ by $-a^2$ and $b$ by $-b^2$ in (1) we get 5+5+5

$$\frac{1}{(x^2+a^2)(x^2+b^2)}=\frac{1}{(-a^2+b^2)}\frac{1}{(x^2+a^2)}-\frac{1}{(-a^2+b^2)}\frac{1}{(x^2+b^2)}\ (15)\ 10$$

$$\therefore\int\frac{1}{(x^2+a^2)(x^2+b^2)}dx=\frac{1}{(b^2-a^2)}\int\frac{1}{(x^2+a^2)}dx-\frac{1}{(b^2-a^2)}\int\frac{1}{(x^2+b^2)}dx\ (05)$$

$$=\frac{1}{(b^2-a^2)}\left\{\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)-\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)\right\}+C \text{ if } a,b\neq 0,$$

10 10 5

where $C$ is an arbitrary constant ~~(20)~~

If $a=0$ then $b\neq 0$.

$$\therefore\int\frac{1}{x^2(x^2+b^2)}dx=\frac{1}{b^2}\int\frac{1}{x^2}dx-\frac{1}{b^2}\int\frac{1}{(x^2+b^2)}dx$$

$$=-\frac{1}{b^2x}-\frac{1}{b^3}\tan^{-1}\left(\frac{x}{b}\right)+C' \text{ where } C' \text{ is an arbitrary constant}$$

(05)

If $b=0$ then $a\neq 0$.

$$\therefore\int\frac{1}{x^2(x^2+b^2)}dx=-\frac{1}{a^2}\int\frac{1}{(x^2+a^2)}dx+\frac{1}{a^2}\int\frac{1}{x^2}dx$$

$$=-\frac{1}{a^2x}-\frac{1}{a^3}\tan^{-1}\left(\frac{x}{a}\right)+C'' \text{ where } C'' \text{ is an arbitrary constant.}$$

(05) (ඕනෑම 1 case) [50] $\boxed{60}$

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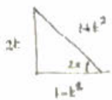
$1=A(x-b)+B(x-a)$
$=(A+B)x-(Ab+Ba)$

$A+B=0$
$Ab+Ba=1$

$A=\dfrac{1}{a-b}$, $B=\dfrac{1}{b-a}$

$t=\tan x$
$\tan 2x=\dfrac{2\tan x}{1-\tan^2 x}=\dfrac{2t}{1-t^2}$




$\cos 2x=$
$\sin 2x=$

16. (a) Show that the equations of the two straight lines drawn through the origin perpendicular to each other so as to form an isosceles right angled triangle with the straight line $lx+my+1=0$ are $(l-m)x+(l+m)y=0$ and $(l-m)x-(l-m)y=0$.

(b) Show that, if the circle $S'\equiv x^2+y^2+2g'x+2f'y+c'=0$ cuts the circle $S\equiv x^2+y^2+2gx+2fy+c=0$ at the ends of a diameter of the circle $S=0$, then $2g^2+2f^2-c=2gg'+2ff'-c'$.

A variable circle cuts each of the circles $S_1\equiv x^2+y^2-25=0$ and $S_2\equiv x^2+y^2-2x-4y-11=0$ at the ends of a diameter. Show that the centre of the variable circle lies on the straight line $x+2y+2=0$.

(a) The equation of a line through the origin can be written as $y=kx$. (05)

If this line intersects the line $lx+my+1=0$ then the $x$ coordinate of the point of intersection satisfies the equation $lx+m(kx)+1=0$ which implies $x=-\dfrac{1}{l+mk}$. ~~(10)~~ 5

The coordinates of the point of intersection of the line $y=kx$ and the line $lx+my+1=0$ are $\left(-\dfrac{1}{l+mk},\ -\dfrac{k}{l+mk}\right)$. (05) + (05)

The distance from the origin to this point of intersection

$$=\sqrt{\left(-\frac{1}{l+mk}\right)^2+\left(-\frac{k}{l+mk}\right)^2}=\frac{\sqrt{1+k^2}}{|l+mk|}$$

(05) (05)

Replacing $k$ by $-\dfrac{1}{k}$ in $\dfrac{\sqrt{1+k^2}}{|l+mk|}$ (05) the distance from the origin to the point of intersection of the perpendicular to the line $y=kx$ and the line $lx+my+1=0$ is obtained

as $\dfrac{\sqrt{1+\left(-\frac{1}{k}\right)^2}}{\left|l+m\left(-\frac{1}{k}\right)\right|}$. (05)

i.e., $\dfrac{\sqrt{1+k^2}}{|kl-m|}$. (05)

Since the triangle formed is isosceles we have $\dfrac{\sqrt{1+k^2}}{|l+mk|}=\dfrac{\sqrt{1+k^2}}{|kl-m|}$. ~~(05)~~ 10

Thus, we have $k=\dfrac{l+m}{l-m}$. (05)

$\therefore\ -\dfrac{1}{k}=-\dfrac{l-m}{l+m}$ (05)

The equations of the two straight lines are $(l-m)x+(l+m)y=0$ and $(l+m)x-(l-m)y=0$. (05) (05)

[70] $\boxed{75}$

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(b) The equation of the common chord of the two circles $S=0$ and $S'=0$ is $2(g-g')x+2(f-f')y+c-c'=0$. ~~(05)~~ $\boxed{10}$

Since the circle $S'=0$ cuts the circle $S=0$ at the ends of a diameter of the circle $S=0$ the centre $(-g,-f)$ of the circle $S=0$ should be on the line $2(g-g')x+2(f-f')y+c-c'=0$. (05)

Thus, we have $2(g-g')(-g)+2(f-f')(-f)+c-c'=0$. (05)

i.e., $2g^2+2f^2-c=2gg'+2ff'-c'$. (05)

$\boxed{25}$ [20]

Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be the equation of the variable circle. (05)

Since the circle $S = 0$ cuts the circle $S_1 = 0$ at the ends of a diameter of the circle $S_1 = 0$ we have

$-(-25) = -c \Rightarrow c = -25$.

$\cancel{(15)}$ 10 (05)

Since the circle $S = 0$ cuts the circle $S_2 = 0$ at the ends of a diameter of the circle $S_2 = 0$ we have

$2(-1)^2 + 2(-2)^2 - (-11) = -2g + 2(-2)f - c$ $\cancel{(15)}$ 10

i.e. $2g + 4f = -c - 21$. (05)

Substituting for $c$ we get $g + 2f = 2$. (05)

$-g + 2(-f) = -2$ (05)

Thus, the centre $(-g, -f)$ of the variable circle lies on the line $x + 2y + 2 = 0$. 50

(05) $\cancel{100}$

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(a) Using the identity $\cos^2\theta + \sin^2\theta = 1$ or otherwise, determine the real constants $a$ and $b$ such that $\cos^6\theta + \sin^6\theta = a + b\cos 4\theta$.

Hence or otherwise,

(i) sketch the graph of $y = 8(\cos^6 x + \sin^6 x)$,

(ii) find the general solution of the equation $\cos^6 x + \sin^6 x = \frac{5}{4} + \frac{1}{2}\sin 4x$.

(b) Solve the equation $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$.

(a) $\cos^2\theta + \sin^2\theta = 1 \Rightarrow (\cos^2\theta + \sin^2\theta)^3 = 1$. (05)

i.e., $\cos^6\theta + 3\cos^4\theta\sin^2\theta + 3\cos^2\theta\sin^4\theta + \sin^6\theta = 1$ (10)

i.e., $\cos^6\theta + 3\cos^2\theta\sin^2\theta(\cos^2\theta + \sin^2\theta) + \sin^6\theta = 1$

i.e., $\cos^6\theta + \frac{3}{4}(2\cos\theta\sin\theta)^2 + \sin^6\theta = 1$ (05)

i.e., $\cos^6\theta + \sin^6\theta = 1 - \frac{3}{8}(2\sin^2 2\theta)$ (05)

i.e., $\cos^6\theta + \sin^6\theta = 1 - \frac{3}{8}(1 - \cos 4\theta)$

i.e., $\cos^6\theta + \sin^6\theta = \frac{5}{8} + \frac{3}{8}\cos 4\theta$ (05)

Thus, we have

$\cos^6\theta + \sin^6\theta = a + b\cos 4\theta$, where $a = \frac{5}{8}$ and $b = \frac{3}{8}$.

(05) (05)

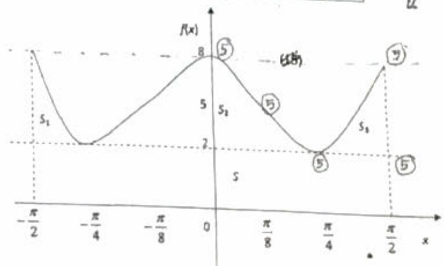
[40]

(i) $f(x) = 8(\sin^6 x + \cos^6 x) = 8\left(\frac{5}{8} + \frac{3}{8}\cos 4x\right) = 5 + 3\cos 4x$ (05)

Drawing the graph of $f(x) = 8(\sin^6 x + \cos^6 x)$ in the range $|x| \le \frac{\pi}{2}$ is the same as drawing the graph of $f(x) = 5 + 3\cos 4x$ in the range $|x| \le \frac{\pi}{2}$.

| $f(x)$ | $-\frac{\pi}{2}$ | $-\frac{\pi}{4}$ | $-\frac{\pi}{8}$ | 0 | $\frac{\pi}{8}$ | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ |
|---|---|---|---|---|---|---|---|
| $5 + 3\cos 4x$ | 8 | 2 | 5 | 8 | 5 | 2 | 8 |

$\cancel{(15)}$




$\cancel{35}$ 30

(ii) $\cos^6 x + \sin^6 x = \frac{5}{4} + \frac{1}{2}\sin 4x$

Since $\cos^6 x + \sin^6 x = \frac{5}{8} + \frac{3}{8}\cos 4x$ we have

$\frac{5}{8} + \frac{3}{8}\cos 4x = \frac{5}{4} + \frac{1}{2}\sin 4x$ (05)

i.e., $3\cos 4x - 4\sin 4x = 5$

i.e., $\frac{3}{5}\cos 4x - \frac{4}{5}\sin 4x = 1$ (05)

i.e., $\cos\alpha\cos 4x - \sin\alpha\sin 4x = 1$, where $\cos\alpha = \frac{3}{5}$ and $\sin\alpha = \frac{4}{5}$.

(05)

i.e., $\cos(4x+\alpha) = 1$ (05)

$\Rightarrow 4x + \alpha = 2n\pi,\ n \in \mathbb{Z}$. (05)

i.e., $x = \frac{n\pi}{2} - \frac{\alpha}{4},\ n \in \mathbb{Z}$ 05/0 [25] $\boxed{30}$

(b) Let $\alpha = \tan^{-1}\left(\frac{x-1}{x-2}\right)$ and $\beta = \tan^{-1}\left(\frac{x+1}{x+2}\right)$. (05)

Then $\tan\alpha = \frac{x-1}{x-2}$, $\tan\beta = \frac{x+1}{x+2}$ and $\alpha + \beta = \frac{\pi}{4}$. (05)

05

$\alpha + \beta = \frac{\pi}{4} \Rightarrow \tan(\alpha+\beta) = 1$ (05)

But $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$ (05)

$$= \frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)} \quad (05)$$

$$= \frac{(x-1)(x+2) + (x+1)(x-2)}{(x^2-4) - (x^2-1)} = \frac{2x^2-4}{-3} \quad \cancel{(10)}\ 5$$

Thus, we have $\frac{2x^2-4}{-3} = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$. (05)

(05) (05) [50]

$\cos^6\theta + \sin^6\theta = (\cos^2\theta)^3 + (\sin^2\theta)^3$

$= 1\,[\cos^4\theta - \cos^2\theta\sin^2\theta + \sin^4\theta]$

$= [1 - 3\cos^2\theta\sin^2\theta]$

$= 1 - \frac{3}{4}(4\sin^2\theta\cos^2\theta)$

$= 1 - \frac{3}{4}\sin^2 2\theta$

$= 1 - \frac{3}{4}\left(\frac{1 - \cos 4\theta}{2}\right)$

$= \frac{5}{8} + \frac{3}{8}\cos 4\theta$

$\cos 2x$

$= \cos^2 x - \sin^2 x$

$= (1 - \tan^2 x)\cos^2 x$

$= \frac{1-\tan^2 x}{\sec^2 x}$

$= \frac{1-t^2}{1+t^2}$

$\sin 2x$

$= 2\tan x\cos^2 x$

$= \frac{2\tan x}{\sec^2 x}$

$= \frac{2t}{1+t^2}$

## G. C. E. (Advanced Level) Examination, August 2011
## Combined Mathematics II
## Scheme of Marking
### Part A

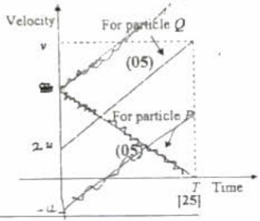
1. A particle $P$ is projected vertically upwards from a point $O$ in space with velocity $2u$. At the same instant, another particle $Q$ is projected vertically downwards from the same point $O$ with the velocity $u$. Both particles move under gravity. Draw the velocity-time graphs for the motions of the particles $P$ and $Q$ in **the same figure** and show that the speed of the particle $Q$ when the particle $P$ reaches its maximum height, is $3u$.

Let $T$ be the time required for the particle $P$ to reach the maximum height.
Let $v$ be the required velocity of the particle $Q$.

Then $\frac{2u}{T} = g \rightarrow (1)$ **(05)**

Also, $\frac{v-u}{T} = g \rightarrow (2)$ **(05)**

From (1) and (2) we get
$v - u = 2u \Rightarrow v = 3u$ **(05)**




[25]

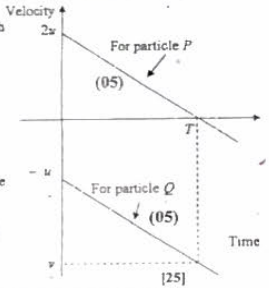
**Aliter:**
Let $T$ be the time required for the particle $P$ to reach the maximum height.
Let $v$ be the required velocity of the particle $Q$.

Then $\frac{2u}{T} = g \rightarrow (1)$ **(05)**

Also, $\frac{-u-V}{T} = g \rightarrow (2)$ **(05)**

From (1) and (2) we get
$-u - v = 2u \Rightarrow v = -3u$ **(05)**

Thus, the speed of the particle $Q$ when the particle reaches the maximum height is $3u$.




[25]

2. One end of a light inextensible string which passes over a smooth fixed pulley carries a particle of mass $2m$. The string passes under a smooth light pulley which carries a particle of mass $m$. The other end of the string is attached to a ceiling as shown in the figure. The system moves freely under gravity. Show that the tension of the string is $\frac{2}{3}mg$.

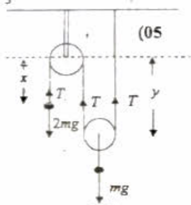
Since the string is inextensible $x + 2y =$ constant.
$\therefore \ddot{x} + 2\ddot{y} = 0 \rightarrow (1)$ **(05)**

Applying $\cancel{P = mf}$ $F = ma$ for the motion of the particle of mass $2m$ vertically upwards we have
$T - 2mg = 2m(-\ddot{x}) \rightarrow (2)$ **(05)**

Applying $\cancel{P = mf}$ $F = ma$ for the motion of the particle of mass $m$ vertically upwards we have
$2T - mg = m(-\ddot{y}) \rightarrow (3)$ **(05)**

$(2) + 4 \times (3) \Rightarrow 9T - 6mg = -2m(\ddot{x} + 2\ddot{y}) = 0$ Form (1) we have

$T = \frac{2}{3}mg$. **(05)**

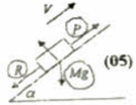



[25]

3. The total mass of a cyclist and his bicycle is $M$ kg. When he rides directly up a straight road inclined at an angle $\alpha$ to the horizontal, at a constant speed of $V$ m s$^{-1}$ against a resistance to motion of $R$ N, he works at a constant rate of $H$ W. Show that $H = (R + Mg\sin\alpha)V$.

Applying $\cancel{P = mf}$ $F = mg$ for the motion of the cyclist along the road upwards we have
$P - R - Mg\sin\alpha = 0$ ~~(10)~~ 0

Also, we have
$H = PV$ **(05)**




$\therefore \frac{H}{V} = R + Mg\sin\alpha \Rightarrow H = (R + Mg\sin\alpha)V$ . **(05)**

[25]

4. A thin light elastic spring of natural length $l$ and modulus of elasticity $\lambda$ rests on a smooth horizontal table. One of its ends is fasten to a fixed point on the table. A particle of mass $m$ is attached to the other end. The spring is stretched along the table and released. Show that the particle performs a simple harmonic motion with periodic time $2\pi\sqrt{\frac{ml}{\lambda}}$.

$T = \lambda\frac{x}{l}$ **(05)**

Applying $\cancel{P = mf}$ $F = ma$ for the motion of the particle of mass $m$ horizontally we have
$-T = m\ddot{x}$ **(05)**

$\therefore \ddot{x} = -\frac{\lambda}{ml}x$ **(05)**


Thus, the particle performs a simple harmonic motion. **(05)**

The periodic time
$= \frac{2\pi}{\sqrt{\lambda/ml}} = 2\pi\sqrt{\frac{ml}{\lambda}}$ . **(05)**

[25]

5. Let $-2\mathbf{p} + 5\mathbf{q}$, $7\mathbf{p} - \mathbf{q}$ and $\mathbf{p} + 3\mathbf{q}$ be the position vectors of three points $A$, $B$ and $C$ respectively, with respect to a fixed origin $O$, where $\mathbf{p}$ and $\mathbf{q}$ are two non-parallel vectors. Show that the points $A$, $B$ and $C$ are collinear and find the ratio in which $C$ divides $AB$.

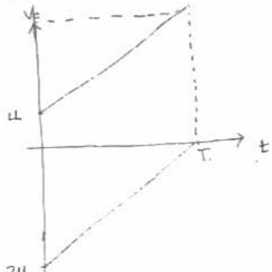
$\overrightarrow{AC} = \mathbf{p} + 3\mathbf{q} - (-2\mathbf{p} + 5\mathbf{q}) = 3\mathbf{p} - 2\mathbf{q}$ . **(05)**

$\overrightarrow{CB} = 7\mathbf{p} - \mathbf{q} - (\mathbf{p} + 3\mathbf{q}) = 6\mathbf{p} - 4\mathbf{q} = 2\overrightarrow{AC}$ . **(05)** 05

$\overrightarrow{AC} = \frac{1}{2}\overrightarrow{CB}$ **(05)**

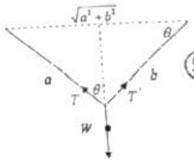
Thus, the points $A$, $B$ and $C$ are collinear and $\cancel{AC = \frac{1}{2}CB}$ $\left\{ AC : CB = 1 : 2,\ \frac{AC}{CB} = \frac{1}{2} \right.$ [25]

**(05)** **(05)**




வேகம் மாறிலி V = 3u

6 A weight $W$ is suspended by two light inextensible strings of lengths $a$ and $b$ from two points at the same horizontal level which are at a distance $\sqrt{a^2+b^2}$ apart. Show that the tensions in the strings are $\frac{Wa}{\sqrt{a^2+b^2}}$ and $\frac{Wb}{\sqrt{a^2+b^2}}$.




5

Resolving vertically we have

$T\cos\theta + T'\sin\theta = W$ (05)

$T\frac{b}{\sqrt{a^2+b^2}} + T'\frac{a}{\sqrt{a^2+b^2}} = W$ ~~(05)~~

$Tb + T'a = W\sqrt{a^2+b^2} \rightarrow (1)$

Resolving horizontally we have

$T\sin\theta - T'\cos\theta = 0$ (05)

$T\frac{a}{\sqrt{a^2+b^2}} - T'\frac{b}{\sqrt{a^2+b^2}} = 0$

$Ta - T'b = \cancel{W\sqrt{a^2+b^2}}\ 0 \rightarrow (2)$

✓ [illegible]

✓ [illegible]

$(1)\times b + (2)\times a \Rightarrow T = \frac{Wa}{\sqrt{a^2+b^2}}$. (05)

$(1)\times a - (2)\times b \Rightarrow T' = \frac{Wb}{\sqrt{a^2+b^2}}$ (05) [25]

---

7. Let $A$ and $B$ be two exhaustive events in a sample space $\Omega$ (that is $A\cup B = \Omega$). If $P(A) = \frac{2}{5}$ and $P(A\cap B) = \frac{1}{3}$, find (i) $P(B)$, (ii) $P(A|B)$, (iii) $P(A'|B')$, where $A'$ and $B'$ are the complementary events of $A$ and $B$ respectively.

(i) $P(\Omega) = P(A\cup B) = P(A) + P(B) - P(A\cap B)$.

$1 = \frac{2}{5} + P(B) - \frac{1}{3} \Rightarrow P(B) = \frac{14}{15}$.

(05) (05)

(ii) $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{\frac{1}{3}}{\frac{14}{15}} = \frac{5}{14}$ (05)

(iii) $P(A'|B') = \frac{P(A'\cap B')}{P(B')} = \frac{P\{(A\cup B)'\}}{P(B')} = \frac{P(\Phi)}{P(B')} = 0 \quad \because P(\Phi) = 0$

(05) (05) [25]

---

8. Two friends attempt independently to solve a problem; their probabilities of success being $\frac{1}{3}$ and $\frac{1}{4}$. Find the probability that (i) both of them, (ii) none of them, will succeed in solving the problem.

$P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{4}$, where $A$ and $B$ are the events that the two friends being success in solving the problem.

(i) $P(A\cap B) = P(A)P(B) = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right) = \frac{1}{12}$.

(05) (05)

(ii) $P\{(A\cup B)'\} = 1 - \{P(A) + P(B) - P(A\cap B)\} = 1 - \left\{\frac{1}{3} + \frac{1}{4} - \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)\right\} = 1 - \frac{1}{2} = \frac{1}{2}$

(05) (05) (05) [25]

---

The daily expenditure of 1000 families is given in the following table:

| Daily expenditure in Rupees | 400 - 600 | 600 - 800 | 800 - 1000 | 1000 - 1200 | 1200 - 1400 |
|---|---|---|---|---|---|
| Number of families | 50 | $x$ | 500 | $y$ | 50 |

If the median of the distribution is 900 Rupees, find the frequencies $x$ and $y$, and show that the mean of the distribution is also 900 Rupees.

Since the median is 900 we have

1000 equal

[illegible]

$50 + x + \frac{500}{200}\times 100 = 500 \Rightarrow x = 200$, $\quad 50 + y + \frac{500}{200}\times 100 = 500 \Rightarrow y = 200$

(05) (05) (05) (05)

Since the distribution is symmetrical the mean is equal to the median.

Thus, the mean is also 900. (05) [25]

---

10. Over the past 15 months, the number of orders received for a certain product has an average of 24 orders per month. The best three months has an average of 35 orders per month. There were 11, 14, 16 and 22 order for the product in the lowest four months.

Find (i) the average of the number of orders received in the remaining 8 months,

(ii) the first quartile of the number of orders of the 15 months.

(i) Total sum $= 24\times 15 = 360$ 5

Total sum of the best three months $= 35\times 3 = 105$

Total sum of the lowest four months $= 11+14+16+22 = 63$

Total of the remaining eight months $= 360 - 105 - 63 = 192$ (05).

Average of the remaining eight months $= \frac{192}{8} = 24$.

~~(05)~~ (05) [15]

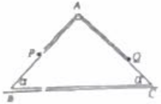
(ii) Since there are 15 data fourth datum should be the first quartile of the distribution. (05)

Thus, 22 is the first quartile of the distribution. (05) [10]

## Part B

11 (a) The top-most points $A$, $B$ and $C$ of three lamp-posts lie in a horizontal plane at the vertices of an equilateral triangle of side $a$. A wind blows in the direction of $\overrightarrow{AC}$ at a steady speed $u$. A bird, whose speed relative to the wind is $v\ (>u)$, flies from $A$ to $B$ along $AB$ and then from $B$ to $C$ along $BC$.
Draw the velocity triangles of relative velocities for both parts of the journey **in the same figure**.
**Hence**, show that the total time taken for the journey from $A$ to $C$ through $B$ is $\dfrac{4a}{u+\sqrt{4v^2-3u^2}}$

(b) A small smooth pulley is fixed at the vertex $A$ of the triangular vertical cross-section $ABC$ of a smooth wedge of mass $2m$ through its centre of mass. The face through $BC$ is placed on a fixed smooth horizontal table. It is given that $AB$ and $AC$ are lines of greatest slope of the relevant faces and $A\hat{B}C = A\hat{C}B = \alpha$. Two smooth particles $P$ and $Q$ of masses $m$ and $\lambda m$ $(\lambda > 1)$ respectively, are attached to the ends of a light inextensible string. The string passes over the pulley and the particles $P$ and $Q$ are placed on $AB$ and $AC$ respectively, with the string taut as shown in the figure.

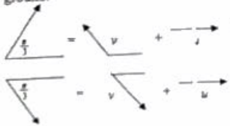



The system is released from rest.
Obtain the equations of motion for the particles $P$ and $Q$ along $BA$ and $AC$ respectively, and for the system horizontally.
Show that the magnitude of the acceleration of each of the particles $P$ and $Q$ relative to the wedge is

$$\frac{(\lambda-1)(\lambda+3)g\sin\alpha}{(\lambda+1)[(\lambda+3)-(\lambda+1)\cos^2\alpha]}$$

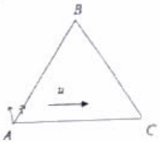
When the particle $Q$ reaches $C$, the string is suddenly broken. Assuming that $P$ has not reached the pulley, write down the magnitude of the acceleration of the particle $P$ relative to the wedge just after the string is broken.

(a) Vel. B, G = Vel. B, W + Vel. W, G , where B for bird, W for wind and G for ground.



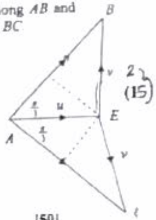

For motion along $AB$

For motion along $BC$




$AEB$ is the velocity triangles of relative velocities for the motion along $AB$ and
$AEC$ is the velocity triangles of relative velocities for the motion along $BC$
The velocity of the bird relative to ground along $AB$ is

$$\sqrt{v^2-u^2\sin^2\frac{\pi}{3}}+u\cos\frac{\pi}{3}=\sqrt{v^2-\frac{3}{4}u^2}+\frac{1}{2}u=\frac{1}{2}\left\{\sqrt{4v^2-3u^2}+u\right\}$$
(10) (05)




2
(15)

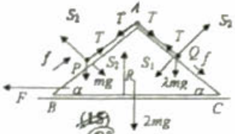
Time taken to fly $AB$ is $\dfrac{2a}{u+\sqrt{4v^2-3u^2}}$ (05)

By symmetry the time taken to fly $BC$ is $\dfrac{2a}{u+\sqrt{4v^2-3u^2}}$ (10)

Thus the total time taken to fly $AB$ and $BC$ is $\dfrac{4a}{u+\sqrt{4v^2-3u^2}}$ (05) [50]

Seperate

(b) Let the acceleration of the particle $P$ relative to the wedge be $f$ along $BA$.
Then the acceleration of the particle $Q$ relative to the wedge is $f$ along $AC$
Let the acceleration of the wedge be $F$ along $CA$




AB $\xrightarrow{}$ (10)

(15) (20)

Forces (10) Acceleration (10)

$\boxed{35}$

CA - (10)

Applying $\overset{F=ma}{\cancel{P=mf}}$ for the motion of the particle $P$ along $BA$ we have
$-mg\sin\alpha+T=m(f-F\cos\alpha)\quad\to(1)\ (15)/0$

Applying $\overset{F=ma}{\cancel{P=mf}}$ for the motion of the particle $Q$ along $AC$ we have
$\lambda mg\sin\alpha-T=\lambda m(f-F\cos\alpha)\quad\to(2)\ (15)/0$ [15]

Applying $\overset{F=ma}{\cancel{P=mf}}$ for the motion of the system horizontally along CB we have
$0=2mF+m(F-f\cos\alpha)+\lambda m(F-f\cos\alpha)\quad\to(3)\ (15)/0$ [15]

$\therefore\ F=\dfrac{1+\lambda}{3+\lambda}f\cos\alpha$ . (05)

$$(1)+(2)\Rightarrow -g(1-\lambda)\sin\alpha=(1+\lambda)f-(1+\lambda)F\cos\alpha \quad (05)$$

$$=(1+\lambda)f\left\{1-\frac{(1+\lambda)}{3+\lambda}\cos^2\alpha\right\} \quad (05)$$

$$=\frac{(1+\lambda)}{(3+\lambda)}\left\{(3+\lambda)-(1+\lambda)\cos^2\alpha\right\}f$$

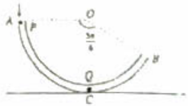
Thus we have $f=\dfrac{(\lambda-1)(3+\lambda)g\sin\alpha}{(1+\lambda)\{(3+\lambda)-(1+\lambda)\cos^2\alpha\}}$ (05)

i.e. the magnitude of the acceleration of the particle $P$ or $Q$ relative to the wedge is
$\dfrac{(1-\lambda)(3+\lambda)g\sin\alpha}{(1+\lambda)\{(3+\lambda)-(1+\lambda)\cos^2\alpha\}}$ ~~(05)~~ ~~[25]~~ $\boxed{20}$

The magnitude of the acceleration of the particle $P$ relative to the wedge just after the string is broken can be found by setting $\lambda=0$ in
$f=\dfrac{(1-\lambda)(3+\lambda)g\sin\alpha}{(1+\lambda)\{(3+\lambda)-(1+\lambda)\cos^2\alpha\}}$ (10) $\boxed{\text{or } T=0}$

Thus we have the magnitude of the acceleration of the particle $P$ relative to the wedge just after the string is broken as $f_1=\dfrac{3g\sin\alpha}{3-\cos^2\alpha}$ ~~(05)~~ (05) [15]

12 A thin smooth tube $ACB$ in the shape of a circular arc of radius $a$ that subtends an angle $\frac{5\pi}{6}$ at its centre $O$ is fixed in a vertical plane with $OA$ horizontal and the lowest point $C$ of the tube touching a fixed horizontal floor as shown in the figure. A smooth particle $P$ of mass $m$ is projected vertically downwards into the tube at the end $A$ with speed $\sqrt{2ga}$.




Show that the speed of the particle $P$, when $OP$ makes an angle $\theta\left(0 \le \theta \le \frac{\pi}{2}\right)$ with $OA$ is $\sqrt{2ga(1+\sin\theta)}$ and the magnitude of the reaction on the particle P from the tube is $mg(2+3\sin\theta)$.

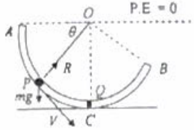
The particle $P$, when it reaches the point $C$, strikes another smooth particle $Q$ of mass $m$ which is at rest inside the tube at $C$. The coefficient of restitution between the particles $P$ and $Q$ is $\frac{1}{2}$.

Find the speed of the particle $P$ just before the collision and show that the speeds of the particles $P$ and $Q$ just after the collision are $\frac{1}{2}\sqrt{ga}$ and $\frac{3}{2}\sqrt{ga}$ respectively.

Show further that the particle $P$ never leaves the tube and that the particle $Q$ reaches the point $B$ with speed $\frac{1}{2}\sqrt{5ga}$.

Find the maximum height from the floor reached by the particle $Q$ after it leaves the tube.

The reaction $R$ is perpendicular to the direction of motion and so it does no work.
Thus, by the conservation of energy for the system we have




$\frac{1}{2}mV^2 - mga\sin\theta = \frac{1}{2}m(2ga)$

(15) / 0

i.e. $V^2 = 2ga(1+\sin\theta)$.

$\therefore V = \sqrt{2ga(1+\sin\theta)}$. (05)

i.e the speed of the particle $P$, when $OP$ makes an angle $\theta$ with $OA$, is $\sqrt{2ga(1+\sin\theta)}$.

~~[20]~~ $\boxed{20}$

Applying $P = mf$ for the motion of the particle $P$ along $PO$ we have

$R - mg\sin\theta = m\frac{V^2}{a} = m\,2g\,a(1+\sin\theta)$. (15)

10 $\therefore R = mg(2+3\sin\theta)$. (05) 5

i.e. the magnitude of the reaction on the particle $P$ from the tube is $mg(2+3\sin\theta)$

[20]

Let $V_1$ be the velocity of the particle $P$ when it reaches the point $C$.

Then taking $\theta = \frac{\pi}{2}$ in $V = \sqrt{2ga(1+\sin\theta)}$ we have

$V_1 = \sqrt{2ga\left(1+\sin\frac{\pi}{2}\right)} = 2\sqrt{ga}$ (10) [10]

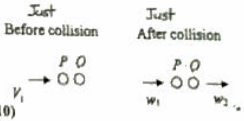
Momentum conservation:

$m2\sqrt{ga} = mw_1 + mw_2$ (10)

i.e. $w_1 + w_2 = 2\sqrt{ga} \to (1)$

Newton's law of restitution:

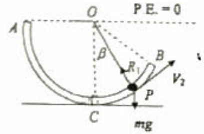
$-(w_1 - w_2) = eV_1 = \frac{1}{2}.2\sqrt{ga} = \sqrt{ga} \to (2)$ (10)




$(1)+(2) \Rightarrow w_2 = \frac{3\sqrt{ga}}{2}$ (05) and $(1)-(2) \Rightarrow w_1 = \frac{\sqrt{ga}}{2}$. (05)

[30]

The reaction $R_1$ is perpendicular to the direction of motion and so it does no work. Thus, by the conservation of energy for the particle $P$ we have




$\frac{1}{2}mV_2^2 - mga\cos\beta = \frac{1}{2}m\,w_1^2 - mga$ (15) 15 / 0

$\frac{1}{2}mV_2^2 - mga\cos\beta = \frac{1}{2}m\left(\frac{ga}{4}\right) - mga$

i.e. $V_2^2 = 2ga\left(\cos\beta - \frac{7}{8}\right)$. (05)

$V_2 = 0 \Leftrightarrow \cos\beta = \frac{7}{8} > \cos\frac{\pi}{3} \Rightarrow \beta < \frac{\pi}{3} \quad \because 0 \le \beta < \frac{\pi}{2}$

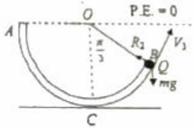
(05) (05) Important

Let this angle be $\beta_0$.

Thus the particle $P$ oscillates between $-\beta_0$ and $\beta_0$. (05)

Hence, the particle $P$ never leaves the tube. (05) [35]

The reaction $R_2$ is perpendicular to the direction of motion and so it does no work. Thus, by the conservation of energy for the particle $Q$ when it reaches the point $B$, we have




$\frac{1}{2}mV_3^2 - mga\cos\frac{\pi}{3} = \frac{1}{2}m\,w_2^2 - mga$ (15) / 0

$\frac{1}{2}mV_3^2 - \frac{1}{2}mga = \frac{1}{2}m\left(\frac{9ga}{4}\right) - mga$

i.e. $V_3^2 = \frac{5ga}{4} \Rightarrow V_3 = \frac{1}{2}\sqrt{5ga}$. (05)

Thus the particle $Q$ reaches the point $B$ with speed $\frac{1}{2}\sqrt{5ga}$. [20]

Applying $v^2 = u^2 + 2fs$ vertically form $B$ to the maximum point it reaches for the motion of the particle $Q$ we have

(10) $0 = V_3^2 \sin^2\frac{\pi}{3} - 2gs = \left(\frac{5ga}{4}\right)\left(\frac{3}{4}\right) - 2gs$, where $s$ is the maximum height the particle $Q$ reaches in its vertical motion.

maximum height

$s = \frac{15}{32}a$. (05)  From the base $= \frac{a}{2} + \frac{15a}{32}$ [15]

$= \frac{31a}{32}$.

13. A particle $P$ of mass $m$ is attached to one end of a light elastic string of natural length $l$. The other end of the string is attached to a fixed point $O$ at a height $4l$ from a horizontal floor. When the particle $P$ hangs in equilibrium, the extension of the string is $l$.

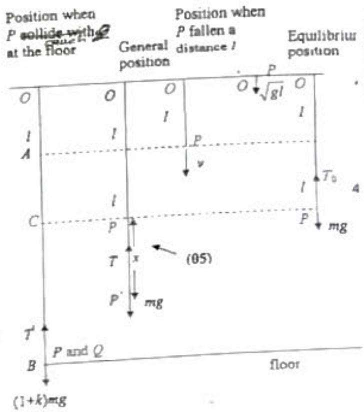
Show that the modulus of elasticity of the string is $mg$.

The particle $P$ is now held at $O$ and projected vertically downwards with a velocity $\sqrt{gl}$. Find the velocity of the particle $P$ when it has fallen a distance $l$.

Write down the equation of motion for the particle $P$, when the length of the string is $2l + x$, where $-l \le x \le 2l$, and show that $\ddot{x} + \frac{g}{l}x = 0$, in the usual notation.

Assuming that the above equation gives $\dot{x}^2 = \frac{g}{l}(c^2 - x^2)$, where $c\,(>0)$ is a constant, find $c$.

Show that the particle $P$ comes to instantaneous rest when it reaches the floor and that the time taken from $O$ to reach the floor is $\frac{1}{3}(3\sqrt{3} - 3 + 2\pi)\sqrt{\frac{l}{g}}$.




Let $T_0$ be the tension in the string in equilibrium position.

Then $T_0 = \frac{\lambda l}{l}$ by Hooke's law.

(10)

But $T_0 = mg$. (10)

Thus we have $\lambda = mg$. (05) [25]

Applying $v^2 = u^2 + 2fs$ vertically downwards to the particle $P$ we have

$v^2 = gl + 2g(l) = 3gl$, (10)

where $v$ is the velocity of $P$ when it has fallen a distance $l$.

Thus we have $v = \sqrt{3gl}$. (05) ~~[20]~~ $\boxed{15}$

Again by Hooke's law we have

$T = \frac{mg(l+x)}{l}$. (10)

Applying Newton's law vertically downwards for the particle $P$ we have

$mg - T = m\ddot{x}$. (10)

i.e. $mg - \frac{mg(l+x)}{l} = m\ddot{x} \Rightarrow \ddot{x} + \frac{g}{l}x = 0$

(05) (05)

~~[30]~~ $\boxed{35}$

any other mtd

$\dot{x} = v = \sqrt{3gl}$, when $x = -l$. (10)

Thus from $\dot{x}^2 = \frac{g}{l}(c^2 - x^2)$ we have

$3gl = \frac{g}{l}(c^2 - l^2) \Rightarrow c = 2l$ [20]

(05) (05)

$\therefore \dot{x}^2 = \frac{g}{l}(4l^2 - x^2)$

$\dot{x} > 0$ for $-l \le x < 2l$ and $\dot{x} = 0$ for $x = 2l$.

(05) (05)

Thus the particle $P$ comes to instantaneous rest when it reaches the floor. (05) [15]



Let $t_1$ be time taken for the particle $P$ to fall under gravity from $O$ to $A$.

Then from $v = u + ft$ we have

$\sqrt{3gl} = \sqrt{gl} + gt_1 \Rightarrow t_1 = (\sqrt{3} - 1)\sqrt{\frac{l}{g}}$

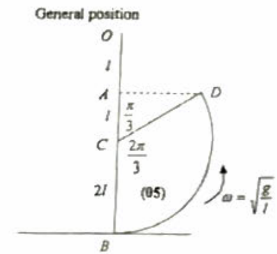
(10) (05)

Let $t_2$ be time taken for the particle $P$ to move in simple harmonic motion from $A$ to $B$.

Then from the figure just above we have

$\sqrt{\frac{g}{l}}\,t_2 = \frac{2\pi}{3} \Rightarrow t_2 = \frac{2\pi}{3}\sqrt{\frac{l}{g}}$

(10) (05)




$t_1 + t_2 = (\sqrt{3} - 1)\sqrt{\frac{l}{g}} + \frac{2\pi}{3}\sqrt{\frac{l}{g}} = \frac{1}{3}(3\sqrt{3} - 3 + 2\pi)\sqrt{\frac{l}{g}}$. (05)

Thus the time taken from $O$ to reach the floor is $\frac{1}{3}(3\sqrt{3} - 3 + 2\pi)\sqrt{\frac{l}{g}}$. [40]

14. (a) Define the dot product $\mathbf{a}\cdot\mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$.

Assuming $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{c}+\mathbf{d})=\mathbf{a}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{c}+\mathbf{a}\cdot\mathbf{d}+\mathbf{b}\cdot\mathbf{d}$ for any four vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$, show that $|\mathbf{a}+\mathbf{b}|^2=|\mathbf{a}|^2+2(\mathbf{a}\cdot\mathbf{b})+|\mathbf{b}|^2$.

Write down a similar expression for $|\mathbf{a}-\mathbf{b}|^2$.

Show that, if $|\mathbf{a}+\mathbf{b}|^2=|\mathbf{a}-\mathbf{b}|^2$ then $\mathbf{a}\cdot\mathbf{b}=0$.

Hence, show that if the diagonals of a parallelogram are equal, then it is a rectangle.

(b) The points $A$, $B$, $C$, $D$, $E$ and $F$ are the vertices of a regular hexagon of side $2a$ metres taken in the anti-clockwise sense. Forces of magnitude $P$, $2P$, $3P$, $4P$, $5P$, $L$, $M$ and $N$ newtons act along $AB$, $CA$, $FC$, $DF$, $ED$, $BC$, $FA$ and $FE$ respectively, in the sense indicated by the order of the letters.

If the system is in equilibrium, find $L$, $M$ and $N$ in terms of $P$.

(a) $\mathbf{a}.\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta$, where $\theta$ is the angle between the two non zero vectors $\mathbf{a}$ and $\mathbf{b}$. (10) [10]

otherwise it is defined to be zero

Taking $\mathbf{c}=\mathbf{a}$ and $\mathbf{d}=\mathbf{b}$ in $(\mathbf{a}+\mathbf{b}).(\mathbf{c}+\mathbf{d})=\mathbf{a}.\mathbf{c}+\mathbf{b}.\mathbf{c}+\mathbf{a}.\mathbf{d}+\mathbf{b}.\mathbf{d}$ we have

$(\mathbf{a}+\mathbf{b}).(\mathbf{a}+\mathbf{b})=\mathbf{a}.\mathbf{a}+\mathbf{b}.\mathbf{a}+\mathbf{a}.\mathbf{b}+\mathbf{b}.\mathbf{b}$ (10)

But $(\mathbf{a}+\mathbf{b}).(\mathbf{a}+\mathbf{b})=|\mathbf{a}+\mathbf{b}||\mathbf{a}+\mathbf{b}|$, $\mathbf{a}.\mathbf{a}=|\mathbf{a}||\mathbf{a}|$, $\mathbf{b}.\mathbf{b}=|\mathbf{b}||\mathbf{b}|$ and $\mathbf{a}.\mathbf{b}=\mathbf{b}.\mathbf{a}$
(05) (05) (05)

$|\mathbf{a}+\mathbf{b}|^2=|\mathbf{a}|^2+2(\mathbf{a}.\mathbf{b})+|\mathbf{b}|^2$. (05) [30]

$|\mathbf{a}-\mathbf{b}|^2=|\mathbf{a}|^2-2(\mathbf{a}.\mathbf{b})+|\mathbf{b}|^2$. (10) [10]

$|\mathbf{a}+\mathbf{b}|^2-|\mathbf{a}-\mathbf{b}|^2=4\,\mathbf{a}.\mathbf{b}$ (05)

If $|\mathbf{a}+\mathbf{b}|^2=|\mathbf{a}-\mathbf{b}|^2$ then $\mathbf{a}.\mathbf{b}=0$. (05) [10]

Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be the position vectors of the vertices $A$, $B$ and $C$ of a parallelogram $OACB$ with respect to the point $O$. (05)

$OC=|\mathbf{c}|$ and $AB=|\mathbf{b}-\mathbf{a}|$.

But $\mathbf{c}=\mathbf{a}+\mathbf{b}$

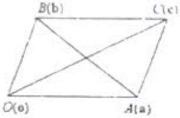
Thus, we have $OC=|\mathbf{a}+\mathbf{b}|$

$OC=AB\Rightarrow|\mathbf{a}+\mathbf{b}|=|\mathbf{a}-\mathbf{b}|$. (05)

From the above result proved we have $\mathbf{a}.\mathbf{b}=0$. (05)

This means that $OA$ is perpendicular to $OB$. (05)

Thus $OABC$ is a rectangle. [20]




(b) Resolving forces horizontally along $\rightarrow$ we have

$$P-2P\cos\frac{\pi}{6}+L\cos\frac{\pi}{3}+3P-4P\cos\frac{\pi}{6}+M\cos\frac{\pi}{3}+N\cos\frac{\pi}{3}+5P=0\ (10)$$

$$9P-3\sqrt{3}P+\frac{1}{2}L+\frac{1}{2}M+\frac{1}{2}N=0$$

i.e. $L+M+N=(\sqrt{3}-3)6P\rightarrow(1)$ (05)

Resolving forces vertically upwards we have

$$-2P\cos\frac{\pi}{3}-4P\cos\frac{\pi}{3}-M\sin\frac{\pi}{3}+N\sin\frac{\pi}{3}+L\sin\frac{\pi}{3}=0\ (10)$$

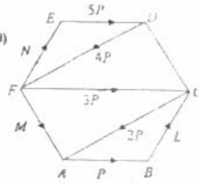
$$-3P-\frac{\sqrt{3}}{2}(M-N-L)=0$$

i.e. $L-M+N=2\sqrt{3}P\rightarrow(2)$ (05)

Taking moment about $F$ in the anti-clockwise we have

$$-2P.2a+P.2a\sin\frac{\pi}{3}-5P.2a\sin\frac{\pi}{3}+L.2.2a\sin\frac{\pi}{3}=0\ (10)$$

$$-2P-2\sqrt{3}P+\sqrt{3}L=0$$




$$L=\frac{2}{\sqrt{3}}(1+\sqrt{3})P\ (10)$$

$(1)-(2)\Rightarrow\ 2M=(\sqrt{3}-3)6P-2\sqrt{3}P=(4\sqrt{3}-18)P\Rightarrow M=(2\sqrt{3}-9)P$ (10)

$(1)+(2)\Rightarrow$

$$2N=(\sqrt{3}-3)6P+2\sqrt{3}P-2L=(8\sqrt{3}-18)P-\frac{4}{\sqrt{3}}(1+\sqrt{3})P=\left(\frac{20\sqrt{3}}{3}-22\right)P$$

$$N=\left(\frac{10}{\sqrt{3}}-11\right)P\ (10)$$

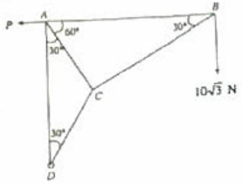
[70]

15. (a) Two uniform rods $AB$ and $BC$ are equal in length. The weight of $AB$ is $2w$ and the weight of $BC$ is $w$. The rods are smoothly hinged at $B$ and the midpoints of the rods are connected by a light inelastic string. The system stands in equilibrium in a vertical plane with $A$ and $C$ on a smooth horizontal table.

If $A\hat{B}C = 2\theta$, show that the tension of the string is $\frac{3}{2}w\tan\theta$.

Find the magnitude of the reaction at $B$ and the angle it makes with the horizontal.

(b) Five light rods $AB$, $BC$, $CD$, $DA$ and $AC$ are smoothly jointed at their ends to form a framework as shown in the figure.



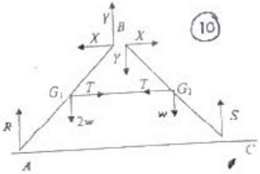

$A\hat{B}C = A\hat{D}C = D\hat{A}C = 30°$ and $B\hat{A}C = 60°$. The framework is smoothly hinged at $D$ and carries a weight of $10\sqrt{3}$ newtons at $B$. The framework is held in a vertical plane, with $AB$ horizontal, by a horizontal force of $P$ newtons at $A$.

(i) Find the value of $P$.

(ii) Find the magnitude and the direction of the reaction at $D$.

(iii) Using Bow's notation, draw a stress diagram for the framework and find the stresses in all the rods, distinguishing between tensions and thrusts.

(a) Let $AB = BC = 2a$.




10

Taking moment about $C$ in the anticlockwise sense for the system we have

$w.a.\sin\theta + 2w.3a\sin\theta - R.4a.\sin\theta = 0$ (15)

$R = \frac{7}{4}w$. (05)

Taking moment about $B$ in the anticlockwise sense for the rod $AB$ we have

$T.a.\cos\theta + 2w.a.\sin\theta - R.2a.\sin\theta = 0$ (15)

$T = -2w\tan\theta + 2R.\tan\theta = \left(-2w + \frac{7}{2}w\right)\tan\theta = \frac{3}{2}w\tan\theta$ (05) ~~[40]~~ 50

Resolving horizontally for the rod $AB$ we have

$X = T = \frac{3}{2}w\tan\theta$ (05)

Resolving vertically for the rod $AB$ we have

$Y + R - 2w = 0$ (05)

$Y = -R + 2w = -\frac{7}{4}w + 2w = \frac{1}{4}w$. ~~(05)~~

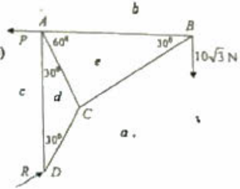
Thus, the reaction at the join $B$

$\sqrt{X^2 + Y^2} = \sqrt{\left(\frac{3}{2}w\tan\theta\right)^2 + \left(\frac{1}{4}w\right)^2} = \frac{w}{4}\sqrt{1 + 36\tan^2\theta}$ (05) ~~[20]~~ 15

The angle that the reaction makes with the horizontal is

$\tan^{-1}\left(\frac{Y}{X}\right) = \tan^{-1}\left(\frac{\frac{w}{4}}{\frac{3}{2}w\tan\theta}\right) = \tan^{-1}\left(\frac{1}{6}\cot\theta\right)$ (10) 5 ~~[10]~~

(b) (i) Taking moments about $D$ we have




$P.AD - 10\sqrt{3}.AB = 0$ (05)

But $AD = 2AC\cos 30°$

$= 2AB\cos 60°\cos 30° = \frac{\sqrt{3}}{2}AB$ (05)

$P.\frac{\sqrt{3}}{2}AB - 10\sqrt{3}.AB = 0$

$P = 20$ N. (05) [15]

Let $R$ be the reaction at $E$ and $\theta$ be the angle that $R$ makes with the horizontal

Resolving forces vertically we have

$R\sin\theta = 10\sqrt{3}$. (05)

Resolving forces horizontally we have $R\cos\theta = P = 20$.

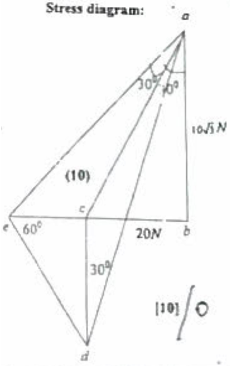
$R = \sqrt{(10\sqrt{3})^2 + 20^2} = 10\sqrt{7}$ N. (05)

$\tan\theta = \frac{10\sqrt{3}}{20} = \frac{\sqrt{3}}{2}$ (05)

$\theta = \tan^{-1}\frac{\sqrt{3}}{2}$

Since the system is in equilibrium under three forces the reaction $R$ should also passes through $B$. [15]

**Stress diagram:**




[10] / 0

| Rod | Stress | Magnitude |
|---|---|---|
| AB | Tension | 30 N |
| BC | Thrust | $20\sqrt{3}$ N |
| AC | Thrust | 20 N |
| DC | Thrust | 40 N |
| AD | Tension | $10\sqrt{3}$ N |

(20) (20) [40]

✓4 → 15
✓3 → 10
✓2 → 05

16 Show that the centre of mass of a uniform solid hemisphere of radius $a$ is on its axis of symmetry at a distance $\frac{3}{8}a$ from the base of the hemisphere.

The inner and outer radii of a uniform solid hemispherical shell are $a$ and $b$ $(> a)$. Show that the distance of its centre of mass from the centre along the axis of symmetry is $\frac{3(a+b)(a^2+b^2)}{8(a^2+ab+b^2)}$

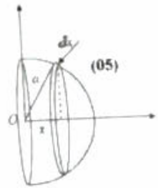
This hemispherical shell rests in equilibrium so that its curved surface is in contact with a rough horizontal ground and equally rough vertical wall.

Show that if the equilibrium is limiting, the inclination of the base to the horizontal is $\sin^{-1}\left\{\frac{8\mu b(1+\mu)(a^2+ab+b^2)}{3(1+\mu^2)(a+b)(a^2+b^2)}\right\}$, where $\mu$ is the coefficient of friction between the shell and the rough surfaces.

By symmetry the centre of mass of the hemisphere lies on the axis of symmetry (05)

Let $\bar{x}$ be the distance to the centre of mass of the hemisphere from O the centre of the base of the hemisphere.

Let $\rho$ be the density of the hemisphere




$\frac{1}{2}\cdot\frac{4}{3}\pi a^3\rho\bar{x}$ (10)

$= \int_0^a \pi(a^2-x^2)x\rho\,dx$ (10)

i.e. $\frac{2}{3}a^3\bar{x} = \int_0^a (a^2x - x^3)dx = \left[a^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^a = \left(\frac{1}{2}-\frac{1}{4}\right)a^4 = \frac{1}{4}a^4 \Rightarrow \bar{x} = \frac{3}{8}a$. (05)

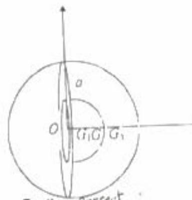
(05) (05)

Thus, the centre of mass of a uniform solid hemisphere of radius $a$ is at a distance $\frac{3}{8}a$ from the base of the hemisphere

[45]

The centre of mass of the solid hemisphere of radius $a$ is at a distance $\frac{3}{8}a$ from $O$. (05)

The centre of mass of the solid hemisphere of radius $b$ is at a distance $\frac{3}{8}b$ from $O$. (05)




Let $\bar{\bar{x}}$ be the distance to the centre of mass of the shell from $O$.

$$\left(\frac{2}{3}\pi b^3 - \frac{2}{3}\pi a^3\right)\rho\bar{\bar{x}} = \left(\frac{2}{3}\pi b^3\right)\rho\frac{3}{8}b - \left(\frac{2}{3}\pi a^3\right)\rho\frac{3}{8}a.$$ ← For the Correct equation 5

(10) (10) (10)

i.e. $\bar{\bar{x}} = \frac{\frac{3}{8}(b^4-a^4)}{(b^3-a^3)} = \frac{3}{8}\frac{(a+b)(a^2+b^2)}{a^2+ab+b^2} \rightarrow$ (1). (05)

05

[45]

Resolving horizontally we have $S = \mu R \rightarrow (1)$ (05)

Resolving vertically we have $R + \mu S = w \rightarrow (2)$ (05)

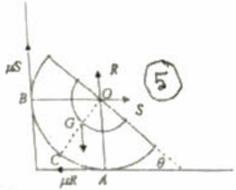
From (1) and (2) we have

$R = \frac{w}{1+\mu^2}$ and $S = \frac{\mu w}{1+\mu^2}$

(05) (05)

Taking moment about $O$ we have

$w OG\sin\theta = \mu R.OA + \mu S.OB$ (15)




i.e. $w\cdot\frac{3}{8}\frac{(a+b)(a^2+b^2)}{a^2+ab+b^2}\sin\theta = \frac{\mu w}{1+\mu^2}b + \frac{\mu^2 w}{1+\mu^2}b$ (10)

i.e. $\frac{3}{8}\frac{(a+b)(a^2+b^2)}{a^2+ab+b^2}\sin\theta = \frac{\mu(1+\mu)}{1+\mu^2}b$ (05)

i.e. $\sin\theta = \frac{8}{3}\frac{\mu b(1+\mu)(a^2+ab+b^2)}{(1+\mu^2)(a+b)(a^2+b^2)}$ (05)

$\theta = \sin^{-1}\left\{\frac{8}{3}\cdot\frac{\mu b(1+\mu)(a^2+ab+b^2)}{(1+\mu^2)(a+b)(a^2+b^2)}\right\}$ (05)

[60]

17. (a) Nimal, Sunil and Piyal play a game with a biased coin which has probability $p$ of landing a head. Nimal, Sunil and Piyal in that order toss the coin in turns. The first person who gets a tail will win the game.

Find the probability that Nimal wins the game in his

(i) second turn,

(ii) third turn.

Hence, find the probability that Nimal wins the game eventually.

Deduce that, if the coin is more likely to land tails than heads, Nimal has more than 50% chance of winning the game.

(b) The mean and the standard deviation of a set of observations $\{x_1, x_2, \ldots, x_n\}$ are $\bar{x}$ and $s_x$ respectively. Suppose that a linear transformation $y_i = a + bx_i$, where $a$ and $b$ are constants transforms the original data set $\{x_1, x_2, \ldots, x_n\}$ to the set $\{y_1, y_2, \ldots, y_n\}$.

Show that $\bar{y} = a + b\bar{x}$ and $s_y^2 = b^2 s_x^2$, where $\bar{y}$ and $s_y$ are the mean and the standard deviation of the set $\{y_1, y_2, \ldots, y_n\}$.

(i) Find the mean and the standard deviation of the set of observations $\{1, 2, 3, 4, 5, 6, 7\}$.

Hence, find

($\alpha$) the mean and the standard deviation of the set of observations $\{2.01, 3.02, 4.03, 5.04, 6.05, 7.06, 8.07\}$.

($\beta$) seven values whose mean is 5 and the standard deviation is 6.

(ii) Salt is packed in bags which the manufacturer claims contain 25 kg each. The following information is given for 80 such bags whose actual weights are not known.

$$\sum_{i=1}^{80}(x_i - 25) = 27.2 \text{ and } \sum_{i=1}^{80}(x_i - 25)^2 = 85.1,$$ where $x_i\,(i = 1, 2, \ldots, 80)$ denotes the actual weight of the $i^{th}$ bag. Using an appropriate linear transformation or otherwise, find the mean and the variance of the actual weights of the eighty bags.

(a) Let $N$ – for Nimal getting a tail, $S$ – for Sunil getting a tail and $P$ – for Piyal getting a tail.

Then $P(N) = P(S) = P(P) = 1 - p = q$ (say) (05) ✓ (5)

$P(N'S'P'N)$ (5)

$ppp(1-p)$ (5) $= p^3(1-p)$ (05) $\boxed{20}$

(iii) P(Nimal wins the game eventually)

$= P(N \cup N'S'P'N \cup \cancel{N'S'P'N'S'P'N} \cup \cdots)$ (05) ✓✓

$= P(N) + P(N'S'P'N) + P(N'S'P'N'S'P'N) + \cdots$ (05) ✓✓

$= P(N) + P(N')P(S')P(P')P(N) + P(N')P(S')P(P')P(N')P(S')P(P')P(N) + \cdots$

(05) (05)

$P(N'S'P'N'S'P'N)$

$= q + p^3q + p^6q + \cdots$ (05) ✓✓

(05) [30] $= p^6(1-p)$ (05)

$= q(1 + p^3 + p^6 + \cdots) = q\dfrac{1}{1-p^3} = \dfrac{1}{1+p+p^2}$ (05) (05) $\boxed{25}$ $\boxed{15}$

(ii) $P(\text{Piyal wins the game})$

$= P(N'S'P \cup N'S'P'N'S'P \cup N'S'P'N'S'P'N'S'P \cup \cdots)$ (05)

$= P(N'S'P) + P(N'S'P'N'S'P) + P(N'S'P'N'S'P'N'S'P) + \cdots$ (05)

$= P(N')P(S')P(P) + P(N')P(S')P(P')P(N')P(S')P(P)$
$+ P(N')P(S')P(P')P(N')P(S')P(P')P(N')P(S')P(P) + \cdots$

$= p^2q + p^5q + p^8q + \cdots$ (05)

$= p^2q(1 + p^3 + p^6 + \cdots) = qp^2\dfrac{1}{1-p^3} = \dfrac{p^2}{1+p+p^2}$. ~~[25]~~

(05) (05)

If the coin is such that it is more likely to lands tails than heads, then $p < \frac{1}{2}$.

Thus, we have $1 + p + p^2 < 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$. (05)



$\therefore$ $P(\text{Nimal wins the game}) = \dfrac{1}{1+p+p^2} > \dfrac{4}{7} = 0.571 > 0.5$

(05) (05)

Nimal has more than 50% chance of wining the game. [15]

(b) $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n}(a + bx_i) = na + b\sum_{i=1}^{n} x_i$ (05)

$\frac{1}{n}\sum_{i=1}^{n} y_i = a + \frac{b}{n}\sum_{i=1}^{n} x_i$

i.e., $\bar{y} = a + b\bar{x}$ (05) [10]

$S_y^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2 = \frac{1}{n}\sum_{i=1}^{n}\{(a + bx_i) - (a + b\bar{x})\}^2 = \frac{b^2}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = b^2S_x^2$

(05) (05) [10]

$\bar{x} = \dfrac{1+2+3+4+5+6+7}{7} = \dfrac{28}{7} = 4$ (05) [05]

$s_x = \sqrt{\frac{1}{7}\{(1-4)^2 + (2-4)^2 + (3-4)^2 + (4-4)^2 + (5-4)^2 + (6-4)^2 + (7-4)^2\}}$

$= \sqrt{\frac{1}{7}\{9+4+1+1+4+9\}} = \sqrt{\frac{28}{7}} = 2$. (05) [05]

(i) $y = 1 + 1.01x$ will transforms the set $\{1, 2, 3, 4, 5, 6, 7\}$ into the set $\{2.01, 3.02, 4.03, 5.04, 6.05, 7.06, 8.07\}$. (05)

$\bar{y} = 1 + 1.01 \times 4 = 5.04$ (05)

$s_y = bs_x = 1.01 \times 2 = 2.02$ (05) [15]

(ii) Consider the transformation $y = a + bx$, where $a$ and $b$ are constants.

Then we have

$\bar{y} = a + b\bar{x}$ and $s_y = bs_x$

Thus we get

$5 = a + 4b$ and $6 = 2b \Rightarrow b = 3$ and $a = -7$.

(05) (05)

$y = -7 + 3x$

The seven numbers are -4, -1, 2, 5, 8, 11, 14. (05) [15]

(c) Consider the transformation $y = -25 + x$. (05)

Then from $\bar{y} = -25 + \bar{x}$ we have $\dfrac{27.2}{80} = -25 + \bar{x} \Rightarrow \bar{x} = 25 + 0.34 = 25.34$

(05) (05) $\boxed{15}$ ~~[10]~~

From $s_y^2 = s_x^2$ we have $s_y = \sqrt{\dfrac{85.1}{80}} = \sqrt{1.06375} = 1.031$

~~(05)~~ (05) ~~[10]~~ $\boxed{05}$